

Algebra 1 Rev - Summer 2011

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Printed: June 20, 2011

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Authors

Andrew Gloag, Anne Gloag, Eve Rawley

Source

Anne Gloag

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Chapter 1

Equations and Functions

1.1 Variable Expressions

Learning Objectives

- Evaluate algebraic expressions.
- Evaluate algebraic expressions with exponents.

Introduction - The Language of Algebra

No one likes doing the same problem over and over again—that’s why mathematicians invented algebra. Algebra takes the basic principles of math and makes them more general, so we can solve a problem once and then use that solution to solve a group of similar problems.

In arithmetic, you’ve dealt with numbers and their arithmetical operations (such as $+$, $-$, \times , \div). In algebra, we use symbols called **variables** (which are usually letters, such as x , y , a , b , c , ...) to represent numbers and sometimes processes.

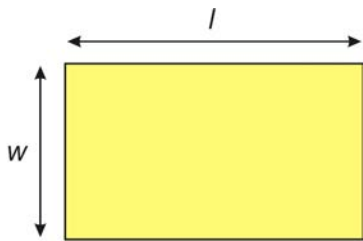
For example, we might use the letter x to represent some number we don’t know yet, which we might need to figure out in the course of a problem. Or we might use two letters, like x and y , to show a relationship between two numbers without needing to know what the actual numbers are. The same letters can represent a wide range of possible numbers, and the same letter may represent completely different numbers when used in two different problems.

Using variables offers advantages over solving each problem “from scratch.” With variables, we can:

- Formulate arithmetical laws such as $a + b = b + a$ for all real numbers a and b .
- Refer to “unknown” numbers. For instance: find a number x such that $3x + 1 = 10$.
- Write more compactly about functional relationships such as, “If you sell x tickets, then your profit will be $3x - 10$ dollars, or “ $f(x) = 3x - 10$,” where “ f ” is the profit function, and x is the input (i.e. how many tickets you sell).

Example 1

Write an algebraic expression for the perimeter and area of the rectangle below.



To find the perimeter, we add the lengths of all 4 sides. We can still do this even if we don't know the side lengths in numbers, because we can use variables like l and w to represent the unknown length and width. If we start at the top left and work clockwise, and if we use the letter P to represent the perimeter, then we can say:

$$P = l + w + l + w$$

We are adding 2 l 's and 2 w 's, so we can say that:

$$P = 2 \cdot l + 2 \cdot w$$

It's customary in algebra to omit multiplication symbols whenever possible. For example, $11x$ means the same thing as $11 \cdot x$ or $11 \times x$. We can therefore also write:

$$P = 2l + 2w$$

Area is *length multiplied by width*. In algebraic terms we get:

$$A = l \times w \rightarrow A = l \cdot w \rightarrow A = lw$$

Note: $2l + 2w$ by itself is an example of a **variable expression**; $P = 2l + 2w$ is an example of an **equation**. The main difference between expressions and **equations** is the presence of an **equals sign** ($=$).

In the above example, we found the simplest possible ways to express the perimeter and area of a rectangle when we don't yet know what its length and width actually are. Now, when we encounter a rectangle whose dimensions we do know, we can simply substitute (or **plug in**) those values in the above equations. In this chapter, we will encounter many expressions that we can evaluate by plugging in values for the variables involved.

Evaluate Algebraic Expressions

When we are given an algebraic expression, one of the most common things we might have to do with it is **evaluate** it for some given value of the variable. The following example illustrates this process.

Example 2

Let $x = 12$. Find the value of $2x - 7$.

To find the solution, we substitute 12 for x in the given expression. Every time we see x , we replace it with 12.

$$\begin{aligned} 2x - 7 &= 2(12) - 7 \\ &= 24 - 7 \\ &= 17 \end{aligned}$$

Note: At this stage of the problem, we place the substituted value in parentheses. We do this to make the written-out problem easier to follow, and to avoid mistakes. (If we didn't use parentheses and also forgot to add a multiplication sign, we would end up turning $2x$ into 212 instead of 2 times 12!)

Example 3

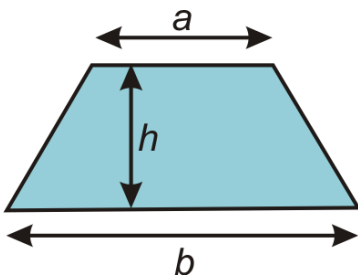
Let $y = -2$. Find the value of $\frac{7}{y} - 11y + 2$.

Solution

$$\begin{aligned}\frac{7}{(-2)} - 11(-2) + 2 &= -3\frac{1}{2} + 22 + 2 \\ &= 24 - 3\frac{1}{2} \\ &= 20\frac{1}{2}\end{aligned}$$

Many expressions have more than one variable in them. For example, the formula for the perimeter of a rectangle in the introduction has two variables: length (l) and width (w). In these cases, be careful to substitute the appropriate value in the appropriate place.

Example 5



The area of a trapezoid is given by the equation $A = \frac{h}{2}(a + b)$. Find the area of a trapezoid with bases $a = 10$ cm and $b = 15$ cm and height $h = 8$ cm.

To find the solution to this problem, we simply take the values given for the variables a , b , and h , and plug them in to the expression for A :

$$\begin{aligned}A &= \frac{h}{2}(a + b) && \text{Substitute 10 for } a, \text{ 15 for } b, \text{ and 8 for } h. \\ A &= \frac{8}{2}(10 + 15) && \text{Evaluate piece by piece. } 10 + 15 = 25; \frac{8}{2} = 4. \\ A &= 4(25) = 100\end{aligned}$$

Solution: The area of the trapezoid is 100 square centimeters.

Evaluate Algebraic Expressions with Exponents

Many formulas and equations in mathematics contain exponents. Exponents are used as a short-hand notation for repeated multiplication. For example:

$$\begin{aligned}2 \cdot 2 &= 2^2 \\ 2 \cdot 2 \cdot 2 &= 2^3\end{aligned}$$

The exponent stands for how many times the number is used as a factor (multiplied). When we deal with integers, it is usually easiest to simplify the expression. We simplify:

$$2^2 = 4$$

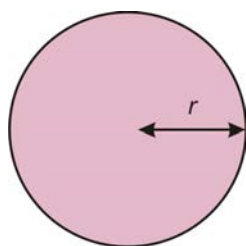
$$2^3 = 8$$

However, we need exponents when we work with variables, because it is much easier to write x^8 than $x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x$.

To evaluate expressions with exponents, substitute the values you are given for each variable and simplify. It is especially important in this case to substitute using parentheses in order to make sure that the simplification is done correctly.

For a more detailed review of exponents and their properties, check out the video at <http://www.mathvids.com/lesson/mathhelp/863-exponents---basics>.

Example 5



The area of a circle is given by the formula $A = \pi r^2$. Find the area of a circle with radius $r = 17$ inches.

Substitute values into the equation.

$$A = \pi r^2 \quad \text{Substitute 17 for } r.$$

$$A = \pi(17)^2 \quad \pi \cdot 17 \cdot 17 \approx 907.9202\dots \text{ Round to 2 decimal places.}$$

The area is approximately 907.92 square inches.

Example 6

Find the value of $\frac{x^2y^3}{x^3+y^2}$, for $x = 2$ and $y = -4$.

Substitute the values of x and y in the following.

$$\frac{x^2y^3}{x^3+y^2} = \frac{(2)^2(-4)^3}{(2)^3+(-4)^2}$$

$$\frac{4(-64)}{8+16} = \frac{-256}{24} = \frac{-32}{3}$$

Substitute 2 for x and -4 for y .

Evaluate expressions: $(2)^2 = (2)(2) = 4$ and

$(2)^3 = (2)(2)(2) = 8$. $(-4)^2 = (-4)(-4) = 16$ and

$(-4)^3 = (-4)(-4)(-4) = -64$.

Example 7

The height (h) of a ball in flight is given by the formula $h = -32t^2 + 60t + 20$, where the height is given in feet and the time (t) is given in seconds. Find the height of the ball at time $t = 2$ seconds.

Solution

$$\begin{aligned}
 h &= -32t^2 + 60t + 20 \\
 &= -32(2)^2 + 60(2) + 20 && \text{Substitute 2 for } t. \\
 &= -32(4) + 60(2) + 20 \\
 &= 12
 \end{aligned}$$

The height of the ball is 12 feet.

Review Questions

- Write the following in a more condensed form by leaving out a multiplication symbol.
 - $2 \times 11x$
 - $1.35 \cdot y$
 - $3 \times \frac{1}{4}$
 - $\frac{1}{4} \cdot z$
- Evaluate the following expressions for $a = -3$, $b = 2$, $c = 5$, and $d = -4$.
 - $2a + 3b$
 - $4c + d$
 - $5ac - 2b$
 - $\frac{2a}{c-d}$
 - $\frac{3b}{d}$
 - $\frac{a-4b}{3c+2d}$
 - $\frac{1}{a+b}$
 - $\frac{ab}{cd}$
- Evaluate the following expressions for $x = -1$, $y = 2$, $z = -3$, and $w = 4$.
 - $8x^3$
 - $\frac{5x^2}{6z^3}$
 - $3z^2 - 5w^2$
 - $x^2 - y^2$
 - $\frac{z^3+w^3}{z^3-w^3}$
 - $2x^3 - 3x^2 + 5x - 4$
 - $4w^3 + 3w^2 - w + 2$
 - $3 + \frac{1}{z^2}$
- The weekly cost C of manufacturing x remote controls is given by the formula $C = 2000 + 3x$, where the cost is given in dollars.
 - What is the cost of producing 1000 remote controls?
 - What is the cost of producing 2000 remote controls?
 - What is the cost of producing 2500 remote controls?
- The volume of a box without a lid is given by the formula $V = 4x(10 - x)^2$, where x is a length in inches and V is the volume in cubic inches.
 - What is the volume when $x = 2$?
 - What is the volume when $x = 3$?

1.2 Order of Operations

Learning Objectives

- Evaluate algebraic expressions with grouping symbols.
- Evaluate algebraic expressions with fraction bars.
- Evaluate algebraic expressions using a graphing calculator.

Introduction

Look at and evaluate the following expression:

$$2 + 4 \times 7 - 1 = ?$$

How many different ways can we interpret this problem, and how many different answers could someone possibly find for it?

The *simplest* way to evaluate the expression is simply to start at the left and work your way across:

$$\begin{aligned} 2 + 4 \times 7 - 1 \\ = 6 \times 7 - 1 \\ = 42 - 1 \\ = 41 \end{aligned}$$

This is the answer you would get if you entered the expression into an ordinary calculator. But if you entered the expression into a scientific calculator or a graphing calculator you would probably get 29 as the answer.

In mathematics, the order in which we perform the various **operations** (such as adding, multiplying, etc.) is important. In the expression above, the operation of **multiplication** takes precedence over **addition**, so we evaluate it first. Let's re-write the expression, but put the multiplication in brackets to show that it is to be evaluated first.

$$2 + (4 \times 7) - 1 = ?$$

First evaluate the brackets: $4 \times 7 = 28$. Our expression becomes:

$$2 + (28) - 1 = ?$$

When we have only addition and subtraction, we start at the left and work across:

$$\begin{aligned} 2 + 28 - 1 \\ = 30 - 1 \\ = 29 \end{aligned}$$

Algebra students often use the word “**PEMDAS**” to help remember the order in which we evaluate the mathematical expressions: **P**arentheses, **E**xponents, **M**ultiplication, **D**ivision, **A**ddition and **S**ubtraction.

Order of Operations

1. Evaluate expressions within **P**arentheses (also all brackets [] and braces { }) first.
2. Evaluate all **E**xponents (terms such as 3^2 or x^3) next.
3. **M**ultiplication *and* **D**ivision is next - work from left to right completing **both** multiplication and division in the order that they appear.
4. Finally, evaluate **A**ddition *and* **S**ubtraction - work from left to right completing **both** addition and subtraction in the order that they appear.

Evaluate Algebraic Expressions with Grouping Symbols

The first step in the order of operations is called **parentheses**, but we include all **grouping symbols** in this step—not just parentheses (), but also square brackets [] and curly braces { }.

Example 1

Evaluate the following:

a) $4 - 7 - 11 + 2$

b) $4 - (7 - 11) + 2$

c) $4 - [7 - (11 + 2)]$

Each of these expressions has the same numbers and the same mathematical operations, in the same order. The placement of the various grouping symbols means, however, that we must evaluate everything in a different order each time. Let's look at how we evaluate each of these examples.

a) This expression doesn't have parentheses, exponents, multiplication, or division. **PEMDAS** states that we treat addition and subtraction as they appear, starting at the left and working right (it's **NOT** addition *then* subtraction).

$$\begin{aligned}4 - 7 - 11 + 2 &= -3 - 11 + 2 \\ &= -14 + 2 \\ &= -12\end{aligned}$$

b) This expression has parentheses, so we first evaluate $7 - 11 = -4$. Remember that when we subtract a negative it is equivalent to adding a positive:

$$\begin{aligned}4 - (7 - 11) + 2 &= 4 - (-4) + 2 \\ &= 8 + 2 \\ &= 10\end{aligned}$$

c) An expression can contain any number of sets of parentheses. Sometimes expressions will have sets of parentheses **inside** other sets of parentheses. When faced with **nested parentheses**, start at the innermost parentheses and work outward.

Brackets may also be used to group expressions which already contain parentheses. This expression has both brackets and parentheses. We start with the innermost group: $11 + 2 = 13$. Then we complete the operation in the brackets.

$$\begin{aligned}4 - [7 - (11 + 2)] &= 4 - [7 - (13)] \\ &= 4 - [-6] \\ &= 10\end{aligned}$$

Example 2

Evaluate the following:

a) $3 \times 5 - 7 \div 2$

b) $3 \times (5 - 7) \div 2$

c) $(3 \times 5) - (7 \div 2)$

a) There are no grouping symbols. **PEMDAS** dictates that we multiply and divide first, working from left to right: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. (NOTE: It's not multiplication *then* division.) Next we subtract:

$$\begin{aligned} 3 \times 5 - 7 \div 2 &= 15 - 3.5 \\ &= 11.5 \end{aligned}$$

b) First, we evaluate the expression inside the parentheses: $5 - 7 = -2$. Then work from left to right:

$$\begin{aligned} 3 \times (5 - 7) \div 2 &= 3 \times (-2) \div 2 \\ &= (-6) \div 2 \\ &= -3 \end{aligned}$$

c) First, we evaluate the expressions inside parentheses: $3 \times 5 = 15$ and $7 \div 2 = 3.5$. Then work from left to right:

$$\begin{aligned} (3 \times 5) - (7 \div 2) &= 15 - 3.5 \\ &= 11.5 \end{aligned}$$

Note that adding parentheses didn't change the expression in part c, but did make it easier to read. Parentheses can be used to change the order of operations in an expression, but they can also be used simply to make it easier to understand.

We can also use the order of operations to simplify an expression that has variables in it, after we substitute specific values for those variables.

Example 3

Use the order of operations to evaluate the following:

a) $2 - (3x + 2)$ when $x = 2$

b) $3y^2 + 2y + 1$ when $y = -3$

c) $2 - (t - 7)^2 \times (u^3 - v)$ when $t = 19$, $u = 4$, and $v = 2$

a) The first step is to substitute the value for x into the expression. We can put it in parentheses to clarify the resulting expression.

$$2 - (3(2) + 2)$$

(Note: $3(2)$ is the same as 3×2 .)

Follow **PEMDAS** - first parentheses. Inside parentheses follow **PEMDAS** again.

$$\begin{aligned} 2 - (3 \times 2 + 2) &= 2 - (6 + 2) && \text{Inside the parentheses, we multiply first.} \\ 2 - 8 &= -6 && \text{Next we add inside the parentheses, and finally we subtract.} \end{aligned}$$

b) The first step is to substitute the value for y into the expression.

$$3 \times (-3)^2 + 2 \times (-3) - 1$$

Follow **PEMDAS**: we cannot simplify the expressions in parentheses, so exponents come next.

$$\begin{array}{ll} 3 \times (-3)^2 + 2 \times (-3) - 1 & \text{Evaluate exponents: } (-3)^2 = 9 \\ = 3 \times 9 + 2 \times (-3) - 1 & \text{Evaluate multiplication: } 3 \times 9 = 27; 2 \times -3 = -6 \\ = 27 + (-6) - 1 & \text{Add and subtract in order from left to right.} \\ = 27 - 6 - 1 & \\ = 20 & \end{array}$$

c) The first step is to substitute the values for t , u , and v into the expression.

$$2 - (19 - 7)^2 \times (4^3 - 2)$$

Follow **PEMDAS**:

$$\begin{array}{ll} 2 - (19 - 7)^2 \times (4^3 - 2) & \text{Evaluate parentheses: } (19 - 7) = 12; (4^3 - 2) = (64 - 2) = 62 \\ = 2 - 12^2 \times 62 & \text{Evaluate exponents: } 12^2 = 144 \\ = 2 - 144 \times 62 & \text{Multiply: } 144 \times 62 = 8928 \\ = 2 - 8928 & \text{Subtract.} \\ = -8926 & \end{array}$$

In parts (b) and (c) we left the parentheses around the negative numbers to clarify the problem. They did not affect the order of operations, but they did help avoid confusion when we were multiplying negative numbers.

Part (c) in the last example shows another interesting point. When we have an expression inside the parentheses, we use **PEMDAS** to determine the order in which we evaluate the contents.

Evaluate Algebraic Expressions with Fraction Bars

Fraction bars count as grouping symbols for **PEMDAS**, so we evaluate them in the first step of solving an expression. All numerators and all denominators can be treated as if they have invisible parentheses around them. When **real** parentheses are also present, remember that the innermost grouping symbols come first. If, for example, parentheses appear on a numerator, they would take precedence over the fraction bar. If the parentheses appear outside of the fraction, then the fraction bar takes precedence.

Example 4

Use the order of operations to evaluate the following expressions:

a) $\frac{z+3}{4} - 1$ when $z = 2$

b) $\left(\frac{a+2}{b+4} - 1\right) + b$ when $a = 3$ and $b = 1$

c) $2 \times \left(\frac{w+(x-2z)}{(y+2)^2} - 1\right)$ when $w = 11$, $x = 3$, $y = 1$, and $z = -2$

a) We substitute the value for z into the expression.

$$\frac{2+3}{4} - 1$$

Although this expression has no parentheses, the fraction bar is also a grouping symbol—it has the same effect as a set of parentheses. We can write in the “invisible parentheses” for clarity:

$$\frac{(2+3)}{4} - 1$$

Using **PEMDAS**, we first evaluate the numerator:

$$\frac{5}{4} - 1$$

We can convert $\frac{5}{4}$ to a mixed number:

$$\frac{5}{4} = 1\frac{1}{4}$$

Then evaluate the expression:

$$\frac{5}{4} - 1 = 1\frac{1}{4} - 1 = \frac{1}{4}$$

b) We substitute the values for a and b into the expression:

$$\left(\frac{3+2}{1+4} - 1\right) + 1$$

This expression has nested parentheses (remember the effect of the fraction bar). The innermost grouping symbol is provided by the fraction bar. We evaluate the numerator ($3 + 2$) and denominator ($1 + 4$) first.

$$\begin{aligned} \left(\frac{3+2}{1+4} - 1\right) + 1 &= \left(\frac{5}{5} - 1\right) - 1 && \text{Next we evaluate the inside of the parentheses. First we divide.} \\ &= (1 - 1) - 1 && \text{Next we subtract.} \\ &= 0 - 1 = -1 \end{aligned}$$

c) We substitute the values for w , x , y , and z into the expression:

$$2 \times \left(\frac{11+(3-2(-2))}{(1+2)^2} - 1\right)$$

This complicated expression has several layers of nested parentheses. One method for ensuring that we start with the innermost parentheses is to use more than one type of parentheses. Working from the outside, we can leave the outermost brackets as parentheses ($$). Next will be the “invisible brackets” from the fraction bar; we will write these as $[]$. The third level of nested parentheses will be the $\{ \}$. We will leave negative numbers in round brackets.

$$\begin{aligned} 2 \times \left(\frac{[11 + \{3 - 2(-2)\}]}{[1 + 2]^2} - 1\right) &&& \text{Start with the innermost grouping sign: } \{ \}. \\ &&& \{1 + 2\} = 3; \{3 - 2(-2)\} = 3 + 4 = 7 \\ &&& \text{Next, evaluate the square brackets.} \\ &= 2 \left(\frac{[11 + 7]}{[3^2]} - 1\right) \\ &= 2 \left(\frac{18}{9} - 1\right) && \text{Next, evaluate the round brackets. Start with division.} \\ &= 2(2 - 1) && \text{Finally, do the addition and subtraction.} \\ &= 2(1) = 2 \end{aligned}$$

Evaluate Algebraic Expressions with a TI-83/84 Family Graphing Calculator

A graphing calculator is a very useful tool in evaluating algebraic expressions. Like a scientific calculator, a graphing calculator follows **PEMDAS**. In this section we will explain two ways of evaluating expressions with the graphing calculator.

Example 5

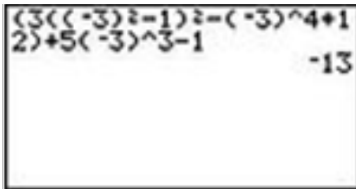
Evaluate $[3(x^2 - 1)^2 - x^4 + 12] + 5x^3 - 1$ when $x = -3$.

Method 1: Substitute for the variable first. Then evaluate the numerical expression with the calculator.

Substitute the value $x = -3$ into the expression.

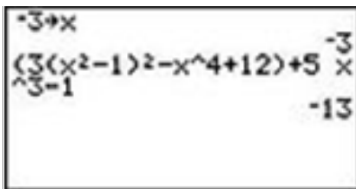
$$[3((-3)^2 - 1)^2 - (-3)^4 + 12] + 5(-3)^3 - 1$$

Input this in the calculator just as it is and press **[ENTER]**. (Note: use \wedge to enter exponents)



The answer is -13.

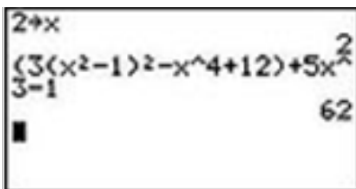
Method 2: Input the original expression in the calculator first and then evaluate.



First, store the value $x = -3$ in the calculator. Type -3 **[STO]** x (The letter x can be entered using the x -**[VAR]** button or **[ALPHA]** + **[STO]**). Then type the original expression in the calculator and press **[ENTER]**.

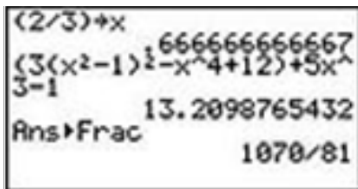
The answer is -13.

The second method is better because you can easily evaluate the same expression for any value you want. For example, let's evaluate the same expression using the values $x = 2$ and $x = \frac{2}{3}$.



For $x = 2$, store the value of x in the calculator: 2 **[STO]** x . Press **[2nd]** **[ENTER]** twice to get the previous expression you typed in on the screen without having to enter it again. Press **[ENTER]** to evaluate the expression.

The answer is 62.



For $x = \frac{2}{3}$, store the value of x in the calculator: $\frac{2}{3}$ [STO] x . Press [2nd] [ENTER] twice to get the expression on the screen without having to enter it again. Press [ENTER] to evaluate.

The answer is 13.21, or $\frac{1070}{81}$ in fraction form.

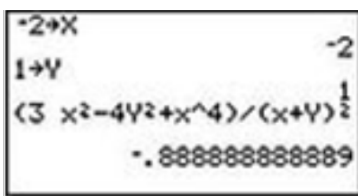
Note: On graphing calculators there is a difference between the minus sign and the negative sign. When we stored the value negative three, we needed to use the negative sign which is to the left of the [ENTER] button on the calculator. On the other hand, to perform the subtraction operation in the expression we used the minus sign. The minus sign is right above the plus sign on the right.

You can also use a graphing calculator to evaluate expressions with more than one variable.

Example 7

Evaluate the expression $\frac{3x^2-4y^2+x^4}{(x+y)^{\frac{1}{2}}}$ for $x = -2$, $y = 1$.

Solution



Store the values of x and y : -2 [STO] x , 1 [STO] y . (The letters x and y can be entered using [ALPHA] + [KEY].) Input the expression in the calculator. When an expression includes a fraction, be sure to use parentheses: $\frac{(\text{numerator})}{(\text{denominator})}$.

Press [ENTER] to obtain the answer $-\overline{.88}$ or $-\frac{8}{9}$.

Additional Resources

For more practice, you can play an algebra game involving order of operations online at <http://www.funbrain.com/algebra/index.html>.

Review Questions

1. Use the order of operations to evaluate the following expressions.

- $8 - (19 - (2 + 5) - 7)$
- $2 + 7 \times 11 - 12 \div 3$
- $(3 + 7) \div (7 - 12)$
- $\frac{2 \cdot (3 + (2 - 1))}{4 - (6 + 2)} - (3 - 5)$
- $\frac{4 + 7(3)}{9 - 4} + \frac{12 - 3 \cdot 2}{2}$
- $(4 - 1)^2 + 3^2 \cdot 2$
- $\frac{(2^2 + 5)^2}{5^2 - 4^2} \div (2 + 1)$

2. Evaluate the following expressions involving variables.

- (a) $\frac{jk}{j+k}$ when $j = 6$ and $k = 12$
- (b) $2y^2$ when $x = 1$ and $y = 5$
- (c) $3x^2 + 2x + 1$ when $x = 5$
- (d) $(y^2 - x)^2$ when $x = 2$ and $y = 1$
- (e) $\frac{x+y^2}{y-x}$ when $x = 2$ and $y = 3$

3. Evaluate the following expressions involving variables.

- (a) $\frac{4x}{9x^2-3x+1}$ when $x = 2$
- (b) $\frac{z^2}{x+y} + \frac{x^2}{x-y}$ when $x = 1$, $y = -2$, and $z = 4$
- (c) $\frac{4xyz}{y^2-x^2}$ when $x = 3$, $y = 2$, and $z = 5$
- (d) $\frac{x^2-z^2}{xz-2x(z-x)}$ when $x = -1$ and $z = 3$

4. Insert parentheses in each expression to make a true equation.

- (a) $5 - 2 \times 6 - 5 + 2 = 5$
- (b) $12 \div 4 + 10 - 3 \times 3 + 7 = 11$
- (c) $22 - 32 - 5 \times 3 - 6 = 30$
- (d) $12 - 8 - 4 \times 5 = -8$

5. Evaluate each expression using a graphing calculator.

- (a) $x^2 + 2x - xy$ when $x = 250$ and $y = -120$
- (b) $(xy - y^4)^2$ when $x = 0.02$ and $y = -0.025$
- (c) $\frac{x+y-z}{xy+yz+xz}$ when $x = \frac{1}{2}$, $y = \frac{3}{2}$, and $z = -1$
- (d) $\frac{(x+y)^2}{4x^2-y^2}$ when $x = 3$ and $y = -5$
- (e) $\frac{(x-y)^3}{x^3-y} + \frac{(x+y)^2}{x+y^4}$ when $x = 4$ and $y = -2$

1.3 Patterns and Equations

Learning Objectives

- Write an equation.
- Use a verbal model to write an equation.
- Solve problems using equations.

Introduction

In mathematics, and especially in algebra, we look for patterns in the numbers we see. The tools of algebra help us describe these patterns with words and with **equations** (formulas or functions). An equation is a mathematical recipe that gives the value of one variable in terms of another.

For example, if a theme park charges \$12 admission, then the number of people who enter the park every day and the amount of money taken in by the ticket office are related mathematically, and we can write a rule to find the amount of money taken in by the ticket office.

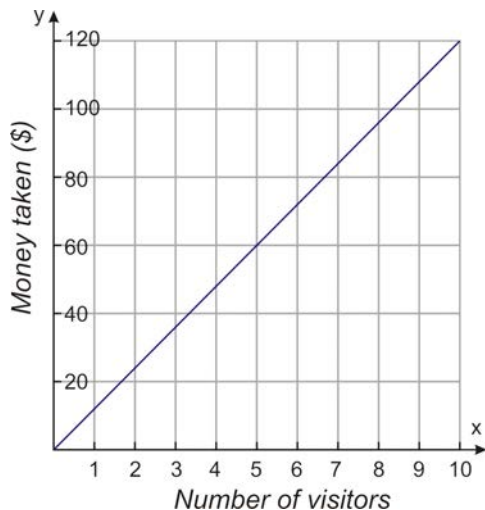
In words, we might say “The amount of money taken in is equal to twelve times the number of people who enter the park.”

We could also make a table. The following table relates the number of people who visit the park and the total money taken in by the ticket office.

Number of visitors	1	2	3	4	5	6	7
Money taken in (\$)	12	24	36	48	60	72	84

Clearly, we would need a **big** table to cope with a busy day in the middle of a school vacation!

A third way we might relate the two quantities (visitors and money) is with a graph. If we plot the money taken in on the **vertical axis** and the number of visitors on the **horizontal axis**, then we would have a graph that looks like the one shown below. Note that this graph shows a smooth line that includes non-whole number values of x (e.g. $x = 2.5$). In real life this would not make sense, because fractions of people can't visit a park. This is an issue of domain and range, something we will talk about later.



The method we will examine in detail in this lesson is closer to the first way we chose to describe the relationship. In words we said that “*The amount of money taken in is twelve times the number of people who enter the park.*” In mathematical terms we can describe this sort of relationship with **variables**. A variable is a letter used to represent an unknown quantity. We can see the beginning of a mathematical formula in the words:

The amount of money taken in **is** twelve **times** the number of people who enter the park.

This can be translated to:

$$\text{the amount of money taken in} = 12 \times (\text{the number of people who enter the park})$$

We can now see which quantities can be assigned to **letters**. First we must state which letters (or **variables**) relate to which quantities. We call this **defining the variables**:

Let x = the number of people who enter the theme park.

Let y = the total amount of money taken in at the ticket office.

We now have a fourth way to describe the relationship: with an algebraic equation.

$$y = 12x$$

Writing a mathematical equation using variables is very convenient. You can perform all of the operations necessary to solve this problem without having to write out the known and unknown quantities over and over again. At the end of the problem, you just need to remember which quantities x and y represent.

Write an Equation

An equation is a term used to describe a collection of **numbers** and **variables** related through mathematical **operators**. An **algebraic equation** will contain letters that represent real quantities. For example, if we wanted to use the algebraic equation in the example above to find the money taken in for a certain number of visitors, we would substitute that number for x and then solve the resulting equation for y .

Example 1

A theme park charges \$12 entry to visitors. Find the money taken in if 1296 people visit the park.

Let's break the solution to this problem down into steps. This will be a useful strategy for all the problems in this lesson.

Step 1: Extract the important information.

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors}) \\ (\text{number of visitors}) &= 1296\end{aligned}$$

Step 2: Translate into a mathematical equation. To do this, we pick variables to stand for the numbers.

$$\begin{aligned}\text{Let } y &= (\text{number of dollars taken in}). \\ \text{Let } x &= (\text{number of visitors}).\end{aligned}$$

$$\begin{aligned}(\text{number of dollars taken in}) &= 12 \times (\text{number of visitors}) \\ y &= 12 \times x\end{aligned}$$

Step 3: Substitute in any known values for the variables.

$$\begin{aligned}y &= 12 \times x \\ x &= 1296 \\ \therefore \\ y &= 12 \times 1296\end{aligned}$$

Step 4: Solve the equation.

$$y = 12 \times 1296 = 15552$$

The amount of money taken in is \$15552.

Step 5: Check the result.

If \$15552 is taken at the ticket office and tickets are \$12, then we can divide the total amount of money collected by the price per individual ticket.

$$(\text{number of people}) = \frac{15552}{12} = 1296$$

1296 is indeed the number of people who entered the park. **The answer checks out.**

Example 2

The following table shows the relationship between two quantities. First, write an equation that describes the relationship. Then, find out the value of b when a is 750.

a	0	10	20	30	40	50
b	20	40	60	80	100	120

Step 1: Extract the important information.

We can see from the table that every time a increases by 10, b increases by 20. However, b is not simply twice the value of a . We can see that when $a = 0$, $b = 20$, and this gives a clue as to what rule the pattern follows. The rule linking a and b is:

“To find b , double the value of a and add 20.”

Step 2: Translate into a mathematical equation:

Table 1.1:

Text	Translates to	Mathematical Expression
“To find b ”	→	$b =$
“double the value of a ”	→	$2a$
“add 20”	→	$+ 20$

Our equation is $b = 2a + 20$.

Step 3: Solve the equation.

The original problem asks for the value of b when a is 750. When a is 750, $b = 2a + 20$ becomes $b = 2(750) + 20$. Following the order of operations, we get:

$$\begin{aligned}
 b &= 2(750) + 20 \\
 &= 1500 + 20 \\
 &= 1520
 \end{aligned}$$

Step 4: Check the result.

In some cases you can check the result by plugging it back into the original equation. Other times you must simply double-check your math. In either case, checking your answer is *always* a good idea. In this case, we can plug our answer for b into the equation, along with the value for a , and see what comes out. $1520 = 2(750) + 20$ is TRUE because both sides of the equation are equal. A true statement means that **the answer checks out**.

Use a Verbal Model to Write an Equation

In the last example we developed a **rule**, written in words, as a way to develop an algebraic **equation**. We will develop this further in the next few examples.

Example 3

The following table shows the values of two related quantities. Write an equation that describes the relationship mathematically.

Table 1.2:

x -value	y -value
-2	10
0	0
2	-10
4	-20
6	-30

Step 1: Extract the important information.

We can see from the table that y is five times bigger than x . The value for y is negative when x is positive, and it is positive when x is negative. Here is the rule that links x and y :

“ y is the negative of five times the value of x ”

Step 2: Translate this statement into a mathematical equation.

Table 1.3:

Text	Translates to	Mathematical Expression
“ y is”	\rightarrow	$y =$
“negative 5 times the value of x ”	\rightarrow	$-5x$

Our equation is $y = -5x$.

Step 3: There is nothing in this problem to **solve** for. We can move to Step 4.

Step 4: Check the result.

In this case, the way we would check our answer is to use the equation to generate our own xy pairs. If they match the values in the table, then we know our equation is correct. We will plug in -2, 0, 2, 4, and 6 for x and solve for y :

Table 1.4:

x	y
-2	$-5(-2) = 10$
0	$-5(0) = 0$
2	$-5(2) = -10$
4	$-5(4) = -20$
6	$-5(6) = -30$

The y -values in this table match the ones in the earlier table. **The answer checks out.**

Example 4

Zarina has a \$100 gift card, and she has been spending money on the card in small regular amounts. She checks the balance on the card weekly and records it in the following table.

Table 1.5:

Week Number	Balance (\$)
1	100
2	78
3	56
4	34

Write an equation for the money remaining on the card in any given week.

Step 1: Extract the important information.

The balance remaining on the card is not just a constant multiple of the week number; 100 is 100 times 1, but 78 is not 100 times 2. But there is still a pattern: the balance decreases by 22 whenever the week number increases by 1. This suggests that the balance is somehow related to the amount “-22 times the week number.”

In fact, the balance equals “-22 times the week number, plus *something*.” To determine what that *something* is, we can look at the values in one row on the table—for example, the first row, where we have a balance of \$100 for week number 1.

Step 2: Translate into a mathematical equation.

First, we define our variables. Let n stand for the week number and b for the balance.

Then we can translate our verbal expression as follows:

Table 1.6:

Text	Translates to	Mathematical Expression
Balance equals -22 times the week number, plus <i>something</i> .	\rightarrow	$b = -22n + ?$

To find out what that $?$ represents, we can plug in the values from that first row of the table, where $b = 100$ and $n = 1$. This gives us $100 = -22(1) + ?$.

So what number gives 100 when you add -22 to it? The answer is 122, so that is the number the $?$ stands for. Now our final equation is:

$$b = -22n + 122$$

Step 3: All we were asked to **find** was the expression. We weren't asked to solve it, so we can move to Step 4.

Step 4: Check the result.

To check that this equation is correct, we see if it really reproduces the data in the table. To do that we plug in values for n :

$$n = 1 \rightarrow b = -22(1) + 122 = 122 - 22 = 100$$

$$n = 2 \rightarrow b = -22(2) + 122 = 122 - 44 = 78$$

$$n = 3 \rightarrow b = -22(3) + 122 = 122 - 66 = 56$$

$$n = 4 \rightarrow b = -22(4) + 122 = 122 - 88 = 34$$

The equation perfectly reproduces the data in the table. **The answer checks out.**

Solve Problems Using Equations

Let's solve the following real-world problem by using the given information to write a mathematical equation that can be solved for a solution.

Example 5

A group of students are in a room. After 25 students leave, it is found that $\frac{2}{3}$ of the original group is left in the room. How many students were in the room at the start?

Step 1: Extract the important information

We know that 25 students leave the room.

We know that $\frac{2}{3}$ of the original number of students are left in the room.

We need to find how many students were in the room at the start.

Step 2: Translate into a mathematical equation. Initially we have an unknown number of students in the room. We can refer to this as the original number.

Let's define the variable x = the original number of students in the room. After 25 students leave the room, the number of students in the room is $x - 25$. We also know that the number of students left is $\frac{2}{3}$ of x . So we have two expressions for the number of students left, and those two expressions are equal because they represent the same number. That means our equation is:

$$\frac{2}{3}x = x - 25$$

Step 3: Solve the equation.

Add 25 to both sides.

$$\begin{aligned}x - 25 &= \frac{2}{3}x \\x - 25 + 25 &= \frac{2}{3}x + 25 \\x &= \frac{2}{3}x + 25\end{aligned}$$

Subtract $\frac{2}{3}x$ from both sides.

$$\begin{aligned}x - \frac{2}{3}x &= \frac{2}{3}x - \frac{2}{3}x + 25 \\ \frac{1}{3}x &= 25\end{aligned}$$

Multiply both sides by 3.

$$\begin{aligned}3 \cdot \frac{1}{3}x &= 3 \cdot 25 \\ x &= 75\end{aligned}$$

Remember that x represents the original number of students in the room. So, there were 75 students in the room to start with.

Step 4: Check the answer:

If we start with 75 students in the room and 25 of them leave, then there are $75 - 25 = 50$ students left in the room.

$\frac{2}{3}$ of the original number is $\frac{2}{3} \cdot 75 = 50$.

This means that the number of students who are left over equals $\frac{2}{3}$ of the original number. **The answer checks out.**

The method of defining variables and writing a mathematical equation is the method you will use the most in an algebra course. This method is often used together with other techniques such as making a table of values, creating a graph, drawing a diagram and looking for a pattern.

Review Questions

Table 1.7:

Day	Profit
1	20
2	40
3	60
4	80
5	100

- The above table depicts the profit in dollars taken in by a store each day.
 - Write a mathematical equation that describes the relationship between the variables in the table.
 - What is the profit on day 10?
 - If the profit on a certain day is \$200, what is the profit on the next day?
- Write a mathematical equation that describes the situation: *A full cookie jar has 24 cookies. How many cookies are left in the jar after you have eaten some?*
 - How many cookies are in the jar after you have eaten 9 cookies?
 - How many cookies are in the jar after you have eaten 9 cookies and then eaten 3 more?
- Write a mathematical equation for the following situations and solve.
 - Seven times a number is 35. What is the number?
 - Three times a number, plus 15, is 24. What is the number?
 - Twice a number is three less than five times another number. Three times the second number is 15. What are the numbers?
 - One number is 25 more than 2 times another number. If each number were multiplied by five, their sum would be 350. What are the numbers?
 - The sum of two consecutive integers is 35. What are the numbers?
 - Peter is three times as old as he was six years ago. How old is Peter?
- How much water should be added to one liter of pure alcohol to make a mixture of 25% alcohol?
- A mixture of 50% alcohol and 50% water has 4 liters of water added to it. It is now 25% alcohol. What was the total volume of the original mixture?
- In Crystal's silverware drawer there are twice as many spoons as forks. If Crystal adds nine forks to the drawer, there will be twice as many forks as spoons. How many forks and how many spoons are in the drawer right now?
- Mia drove to Javier's house at 40 miles per hour. Javier's house is 20 miles away. Mia arrived at Javier's house at 2:00 pm. What time did she leave?
 - Mia left Javier's house at 6:00 pm to drive home. This time she drove 25% faster. What time did she arrive home?

- (c) The next day, Mia took the expressway to Javier's house. This route was 24 miles long, but she was able to drive at 60 miles per hour. How long did the trip take?
- (d) When Mia took the same route back, traffic on the expressway was 20% slower. How long did the return trip take?
8. The price of an mp3 player decreased by 20% from last year to this year. This year the price of the player is \$120. What was the price last year?
9. SmartCo sells deluxe widgets for \$60 each, which includes the cost of manufacture plus a 20% markup. What does it cost SmartCo to manufacture each widget?
10. Jae just took a math test with 20 questions, each worth an equal number of points. The test is worth 100 points total.
- (a) Write an equation relating the number of questions Jae got right to the total score he will get on the test.
- (b) If a score of 70 points earns a grade of $C-$, how many questions would Jae need to get right to get a $C-$ on the test?
- (c) If a score of 83 points earns a grade of B , how many questions would Jae need to get right to get a B on the test?
- (d) Suppose Jae got a score of 60% and then was allowed to retake the test. On the retake, he got all the questions right that he got right the first time, and also got half the questions right that he got wrong the first time. What is his new score?

1.4 Equations and Inequalities

Learning Objectives

- Write equations and inequalities.
- Check solutions to equations.
- Check solutions to inequalities.
- Solve real-world problems using an equation.

Introduction

In algebra, an **equation** is a mathematical expression that contains an equals sign. It tells us that two expressions represent the same number. For example, $y = 12x$ is an equation. An **inequality** is a mathematical expression that contains inequality signs. For example, $y \leq 12x$ is an inequality. Inequalities are used to tell us that an expression is either larger or smaller than another expression. Equations and inequalities can contain both **variables** and **constants**.

Variables are usually given a letter and they are used to represent unknown values. These quantities can change because they depend on other numbers in the problem.

Constants are quantities that remain unchanged. Ordinary numbers like 2, -3 , $\frac{3}{4}$, and π are constants.

Equations and inequalities are used as a shorthand notation for situations that involve numerical data. They are very useful because most problems require several steps to arrive at a solution, and it becomes tedious to repeatedly write out the situation in words.

Write Equations and Inequalities

Here are some examples of equations:

$$3x - 2 = 5 \quad x + 9 = 2x + 5 \quad \frac{x}{3} = 15 \quad x^2 + 1 = 10$$

To write an inequality, we use the following symbols:

$>$ **greater than**

\geq **greater than or equal to**

$<$ **less than**

\leq **less than or equal to**

\neq **not equal to**

Here are some examples of inequalities:

$$3x < 5 \quad 4 - x \leq 2x \quad x^2 + 2x - 1 > 0 \quad \frac{3x}{4} \geq \frac{x}{2} - 3$$

The most important skill in algebra is the ability to translate a word problem into the correct equation or inequality so you can find the solution easily. The first two steps are **defining the variables** and **translating** the word problem into a mathematical equation.

Defining the variables means that we assign letters to any unknown quantities in the problem.

Translating means that we change the word expression into a mathematical expression containing variables and mathematical operations with an equal sign or an inequality sign.

Example 1

Define the variables and translate the following expressions into equations.

- A number plus 12 is 20.
- 9 less than twice a number is 33.
- \$20 was one quarter of the money spent on the pizza.

Solution

a) Define

Let n = the number we are seeking.

Translate

A number plus 12 is 20.

$$n + 12 = 20$$

b) Define

Let n = the number we are seeking.

Translate

9 less than twice a number is 33.

This means that twice the number, minus 9, is 33.

$$2n - 9 = 33$$

c) Define

Let m = the money spent on the pizza.

Translate

\$20 was one quarter of the money spent on the pizza.

$$20 = \frac{1}{4}m$$

Often word problems need to be reworded before you can write an equation.

Example 2

Find the solution to the following problems.

- a) Shyam worked for two hours and packed 24 boxes. How much time did he spend on packing one box?
- b) After a 20% discount, a book costs \$12. How much was the book before the discount?

Solution

- a) **Define**

Let t = time it takes to pack one box.

Translate

Shyam worked for two hours and packed 24 boxes. This means that two hours is 24 times the time it takes to pack one box.

$$2 = 24t$$

Solve

$$t = \frac{2}{24} = \frac{1}{12} \text{ hours}$$
$$\frac{1}{12} \times 60 \text{ minutes} = 5 \text{ minutes}$$

Answer

Shyam takes 5 minutes to pack a box.

- b) **Define**

Let p = the price of the book before the discount.

Translate

After a 20% discount, the book costs \$12. This means that the price minus 20% of the price is \$12.

$$p - 0.20p = 12$$

Solve

$$p - 0.20p = 0.8p, \text{ so } 0.8p = 12$$
$$p = \frac{12}{0.8} = 15$$

Answer

The price of the book before the discount was \$15.

Check

If the original price was \$15, then the book was discounted by 20% of \$15, or \$3. $\$15 - 3 = \12 . **The answer checks out.**

Example 3

Define the variables and translate the following expressions into inequalities.

- The sum of 5 and a number is less than or equal to 2.
- The distance from San Diego to Los Angeles is less than 150 miles.
- Diego needs to earn more than an 82 on his test to receive a *B* in his algebra class.
- A child needs to be 42 inches or more to go on the roller coaster.

Solution

a) Define

Let n = the unknown number.

Translate

$$5 + n \leq 2$$

b) Define

Let d = the distance from San Diego to Los Angeles in miles.

Translate

$$d < 150$$

c) Define

Let x = Diego's test grade.

Translate

$$x > 82$$

d) Define

Let h = the height of child in inches.

Translate:

$$h \geq 42$$

Check Solutions to Equations

You will often need to check solutions to equations in order to check your work. In a math class, checking that you arrived at the correct solution is very good practice. We check the solution to an equation by replacing the variable in an equation with the value of the solution. A solution should result in a true statement when plugged into the equation.

Example 4

Check that the given number is a solution to the corresponding equation.

a) $y = -1$; $3y + 5 = -2y$

b) $z = 3$; $z^2 + 2z = 8$

c) $x = -\frac{1}{2}$; $3x + 1 = x$

Solution

Replace the variable in each equation with the given value.

a)

$$\begin{aligned} 3(-1) + 5 &= -2(-1) \\ -3 + 5 &= 2 \\ 2 &= 2 \end{aligned}$$

This is a true statement. This means that $y = -1$ is a solution to $3y + 5 = -2y$.

b)

$$\begin{aligned} 3^2 + 2(3) &= 8 \\ 9 + 6 &= 8 \\ 15 &= 8 \end{aligned}$$

This is not a true statement. This means that $z = 3$ is **not a solution** to $z^2 + 2z = 8$.

c)

$$\begin{aligned} 3\left(-\frac{1}{2}\right) + 1 &= -\frac{1}{2} \\ \left(-\frac{3}{2}\right) + 1 &= -\frac{1}{2} \\ -\frac{1}{2} &= -\frac{1}{2} \end{aligned}$$

This is a true statement. This means that $x = -\frac{1}{2}$ is a solution to $3x + 1 = x$.

Check Solutions to Inequalities

To check the solution to an inequality, we replace the variable in the inequality with the value of the solution. A solution to an inequality produces a true statement when substituted into the inequality.

Example 5

Check that the given number is a solution to the corresponding inequality.

a) $a = 10$; $20a \leq 250$

b) $b = -0.5$; $\frac{3-b}{b} > -4$

c) $x = \frac{3}{4}$; $4x + 5 \leq 8$

Solution

Replace the variable in each inequality with the given value.

a)

$$\begin{aligned} 20(10) &\leq 250 \\ 200 &\leq 250 \end{aligned}$$

This statement is true. This means that $a = 10$ is a solution to the inequality $20a \leq 250$.

Note that $a = 10$ is not the only solution to this inequality. If we divide both sides of the inequality by 20, we can write it as $a \leq 12.5$. This means that any number less than or equal to 12.5 is also a solution to the inequality.

b)

$$\begin{aligned}\frac{3 - (-0.5)}{(-0.5)} &> -4 \\ \frac{3 + 0.5}{-0.5} &> -4 \\ -\frac{3.5}{0.5} &> -4 \\ -7 &> -4\end{aligned}$$

This statement is false. This means that $b = -0.5$ is not a solution to the inequality $\frac{3-b}{b} > -4$.

c)

$$\begin{aligned}4\left(\frac{3}{4}\right) + 5 &\geq 8 \\ 3 + 5 &\geq 8 \\ 8 &\geq 8\end{aligned}$$

This statement is true. It is true because this inequality includes an equals sign; since 8 is equal to itself, it is also “greater than or equal to” itself. This means that $x = \frac{3}{4}$ is a solution to the inequality $4x + 5 \leq 8$.

Solve Real-World Problems Using an Equation

Let’s use what we have learned about defining variables, writing equations and writing inequalities to solve some real-world problems.

Example 6

Tomatoes cost \$0.50 each and avocados cost \$2.00 each. Anne buys six more tomatoes than avocados. Her total bill is \$8. How many tomatoes and how many avocados did Anne buy?

Solution

Define

Let a = the number of avocados Anne buys.

Translate

Anne buys six more tomatoes than avocados. This means that $a + 6$ = the number of tomatoes.

Tomatoes cost \$0.50 each and avocados cost \$2.00 each. Her total bill is \$8. This means that .50 times the number of tomatoes plus 2 times the number of avocados equals 8.

$$\begin{aligned}0.5(a + 6) + 2a &= 8 \\ 0.5a + 0.5 \cdot 6 + 2a &= 8 \\ 2.5a + 3 &= 8 \\ 2.5a &= 5 \\ a &= 2\end{aligned}$$

Remember that a = the number of avocados, so Anne buys two avocados. The number of tomatoes is $a + 6 = 2 + 6 = 8$.

Answer

Anne bought 2 avocados and 8 tomatoes.

Check

If Anne bought two avocados and eight tomatoes, the total cost is: $(2 \times 2) + (8 \times 0.5) = 4 + 4 = 8$. **The answer checks out.**

Example 7

To organize a picnic Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs. What is the possible number of hamburgers Peter has?

Solution

Define

Let x = number of hamburgers

Translate

Peter needs at least two times as many hamburgers as hot dogs. He has 24 hot dogs.

This means that twice the number of hot dogs is less than or equal to the number of hamburgers.

$$2 \times 24 \leq x, \text{ or } 48 \leq x$$

Answer

Peter needs at least 48 hamburgers.

Check

48 hamburgers is twice the number of hot dogs. So more than 48 hamburgers is more than twice the number of hot dogs. **The answer checks out.**

Additional Resources

For more practice solving inequalities, check out <http://www.aaastudy.com/equ725x7.htm>.

Review Questions

1. Define the variables and translate the following expressions into equations.
 - (a) Peter's Lawn Mowing Service charges \$10 per job and \$0.20 per square yard. Peter earns \$25 for a job.
 - (b) Renting the ice-skating rink for a birthday party costs \$200 plus \$4 per person. The rental costs \$324 in total.
 - (c) Renting a car costs \$55 per day plus \$0.45 per mile. The cost of the rental is \$100.
 - (d) Nadia gave Peter 4 more blocks than he already had. He already had 7 blocks.
2. Define the variables and translate the following expressions into inequalities.
 - (a) A bus can seat 65 passengers or fewer.
 - (b) The sum of two consecutive integers is less than 54.
 - (c) The product of a number and 3 is greater than 30.
 - (d) An amount of money is invested at 5% annual interest. The interest earned at the end of the year is greater than or equal to \$250.
 - (e) You buy hamburgers at a fast food restaurant. A hamburger costs \$0.49. You have at most \$3 to spend. Write an inequality for the number of hamburgers you can buy.

- (f) Mariel needs at least 7 extra credit points to improve her grade in English class. Additional book reports are worth 2 extra credit points each. Write an inequality for the number of book reports Mariel needs to do.
3. Check whether the given number is a solution to the corresponding equation.
- $a = -3$; $4a + 3 = -9$
 - $x = \frac{4}{3}$; $\frac{3}{4}x + \frac{1}{2} = \frac{3}{2}$
 - $y = 2$; $2.5y - 10.0 = -5.0$
 - $z = -5$; $2(5 - 2z) = 20 - 2(z - 1)$
4. Check whether the given number is a solution to the corresponding inequality.
- $x = 12$; $2(x + 6) \leq 8x$
 - $z = -9$; $1.4z + 5.2 > 0.4z$
 - $y = 40$; $-\frac{5}{2}y + \frac{1}{2} < -18$
 - $t = 0.4$; $80 \geq 10(3t + 2)$
5. The cost of a Ford Focus is 27% of the price of a Lexus GS 450h. If the price of the Ford is \$15000, what is the price of the Lexus?
6. On your new job you can be paid in one of two ways. You can either be paid \$1000 per month plus 6% commission of total sales or be paid \$1200 per month plus 5% commission on sales over \$2000. For what amount of sales is the first option better than the second option? Assume there are always sales over \$2000.
7. A phone company offers a choice of three text-messaging plans. Plan A gives you unlimited text messages for \$10 a month; Plan B gives you 60 text messages for \$5 a month and then charges you \$0.05 for each additional message; and Plan C has no monthly fee but charges you \$0.10 per message.
- If m is the number of messages you send per month, write an expression for the monthly cost of each of the three plans.
 - For what values of m is Plan A cheaper than Plan B?
 - For what values of m is Plan A cheaper than Plan C?
 - For what values of m is Plan B cheaper than Plan C?
 - For what values of m is Plan A the cheapest of all? (Hint: for what values is A both cheaper than B and cheaper than C?)
 - For what values of m is Plan B the cheapest of all? (Careful—for what values is B cheaper than A?)
 - For what values of m is Plan C the cheapest of all?
 - If you send 30 messages per month, which plan is cheapest?
 - What is the cost of each of the three plans if you send 30 messages per month?

1.5 Functions as Rules and Tables

Learning Objectives

- Identify the domain and range of a function.
- Make a table for a function.
- Write a function rule.
- Represent a real-world situation with a function.

Introduction

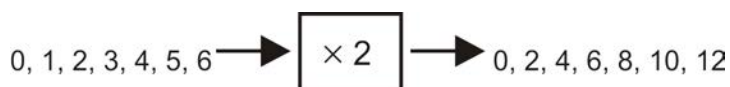
A **function** is a rule for relating two or more variables. For example, the price you pay for phone service may depend on the number of minutes you talk on the phone. We would say that the cost of phone service is a *function* of the number of minutes you talk. Consider the following situation.

Josh goes to an amusement park where he pays \$2 per ride.

There is a relationship between the number of rides Josh goes on and the total amount he spends that day: To figure out the amount he spends, we multiply the number of rides by two. This rule is an example of a **function**. Functions usually—*but not always*—are rules based on mathematical operations. You can think of a function as a box or a machine that contains a mathematical operation.



Whatever number we feed into the function box is changed by the given operation, and a new number comes out the other side of the box. When we input different values for the number of rides Josh goes on, we get different values for the amount of money he spends.

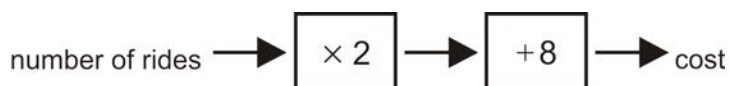


The input is called the **independent variable** because its value can be any number. The output is called the **dependent variable** because its value depends on the input value.

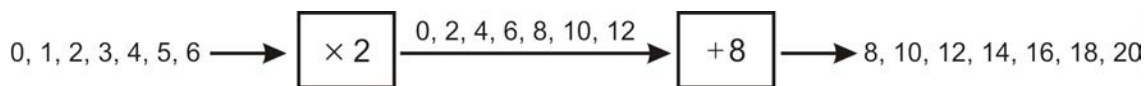
Functions usually contain more than one mathematical operation. Here is a situation that is slightly more complicated than the example above.

Jason goes to an amusement park where he pays \$8 admission and \$2 per ride.

The following function represents the total amount Jason pays. The rule for this function is "multiply the number of rides by 2 and add 8."



When we input different values for the number of rides, we arrive at different outputs (costs).



These flow diagrams are useful in visualizing what a function is. However, they are cumbersome to use in practice. In algebra, we use the following short-hand notation instead:

$$\begin{array}{c} \textit{input} \\ \downarrow \\ \underbrace{f(x)} = y \leftarrow \textit{output} \\ \textit{function} \\ \textit{box} \end{array}$$

First, we define the variables:

x = the number of rides Jason goes on

y = the total amount of money Jason spends at the amusement park.

So, x represents the input and y represents the output. The notation $f()$ represents the function or the mathematical operations we use on the input to get the output. In the last example, the cost is 2 times the number of rides plus 8. This can be written as a function:

$$f(x) = 2x + 8$$

In algebra, the notations y and $f(x)$ are typically used interchangeably. Technically, though, $f(x)$ represents the function itself and y represents the output of the function.

Identify the Domain and Range of a Function

In the last example, we saw that we can input the number of rides into the function to give us the total cost for going to the amusement park. The set of all values that we can use for the input is called the **domain** of the function, and the set of all values that the output could turn out to be is called the **range** of the function. In many situations the **domain** and **range** of a function are both simply the set of all real numbers, but this isn't always the case. Let's look at our amusement park example.

Example 1

Find the domain and range of the function that describes the situation:

Jason goes to an amusement park where he pays \$8 admission and \$2 per ride.

Solution

Here is the function that describes this situation:

$$f(x) = 2x + 8 = y$$

In this function, x is the number of rides and y is the total cost. To find the domain of the function, we need to determine which numbers make sense to use as the input (x).

- The values have to be zero or positive, because Jason can't go on a negative number of rides.
- The values have to be integers because, for example, Jason could not go on 2.25 rides.
- Realistically, there must be a maximum number of rides that Jason can go on because the park closes, he runs out of money, etc. However, since we aren't given any information about what that maximum might be, we must consider that all non-negative integers are possible values regardless of how big they are.

Answer For this function, the domain is the set of all non-negative integers.

To find the range of the function we must determine what the values of y will be when we apply the function to the input values. The domain is the set of all non-negative integers: $\{0, 1, 2, 3, 4, 5, 6, \dots\}$. Next we plug these values into the function for x . If we plug in 0, we get 8; if we plug in 1, we get 10; if we plug in 2, we get 12, and so on, counting by 2s each time. Possible values of y are therefore 8, 10, 12, 14, 16, 18, 20... or in other words all even integers greater than or equal to 8.

Answer The range of this function is the set of all even integers greater than or equal to 8.

Example 2

Find the domain and range of the following functions.

a) A ball is dropped from a height and it bounces up to 75% of its original height.

b) $y = x^2$

Solution

a) Let's define the variables:

x = original height

y = bounce height

A function that describes the situation is $y = f(x) = 0.75x$. x can represent any real value greater than zero, since you can drop a ball from any height greater than zero. A little thought tells us that y can also represent any real value greater than zero.

Answer

The domain is the set of all real numbers greater than zero. The range is also the set of all real numbers greater than zero.

b) Since there is no word problem attached to this equation, we can assume that we can use any real number as a value of x . When we square a real number, we always get a non-negative answer, so y can be any non-negative real number.

Answer

The domain of this function is all real numbers. The range of this function is all non-negative real numbers.

In the functions we've looked at so far, x is called the **independent variable** because it can be any of the values from the domain, and y is called the **dependent variable** because its value depends on x . However, any letters or symbols can be used to represent the dependent and independent variables. Here are three different examples:

$$y = f(x) = 3x$$

$$R = f(w) = 3w$$

$$v = f(t) = 3t$$

These expressions all represent the same function: a function where the dependent variable is three times the independent variable. Only the symbols are different. In practice, we usually pick symbols for the dependent and independent variables based on what they represent in the real world—like t for time, d for distance, v for velocity, and so on. But when the variables don't represent anything in the real world—or even sometimes when they do—we traditionally use y for the dependent variable and x for the independent variable.

For another look at the domain of a function, see the following video, where the narrator solves a sample problem from the California Standards Test about finding the domain of an unusual function: <http://www.youtube.com/watch?v=NRB6s77nx2gI>.

Make a Table For a Function

A table is a very useful way of arranging the data represented by a function. We can match the input and output values and arrange them as a table. For example, the values from Example 1 above can be arranged in a table as follows:

x	0	1	2	3	4	5	6
y	8	10	12	14	16	18	20

A table lets us organize our data in a compact manner. It also provides an easy reference for looking up data, and it gives us a set of coordinate points that we can plot to create a graph of the function.

Example 3

Make a table of values for the function $f(x) = \frac{1}{x}$. Use the following numbers for input values: -1, -0.5, -0.2, -0.1, -0.01, 0.01, 0.1, 0.2, 0.5, 1.

Solution

Make a table of values by filling the first row with the input values and the next row with the output values calculated using the given function.

x	-1	-0.5	-0.2	-0.1	-0.01	0.01	0.1	0.2	0.5	1
$f(x) = \frac{1}{x}$	$\frac{1}{-1}$	$\frac{1}{-0.5}$	$\frac{1}{-0.2}$	$\frac{1}{-0.1}$	$\frac{1}{-0.01}$	$\frac{1}{0.01}$	$\frac{1}{0.1}$	$\frac{1}{0.2}$	$\frac{1}{0.5}$	$\frac{1}{1}$
y	-1	-2	-5	-10	-100	100	10	5	2	1

When you're given a function, you won't usually be told what input values to use; you'll need to decide for yourself what values to pick based on what kind of function you're dealing with. We will discuss how to pick input values throughout this book.

Write a Function Rule

In many situations, we collect data by conducting a survey or an experiment, and then organize the data in a table of values. Most often, we want to find the function rule or formula that fits the set of values in the table, so we can use the rule to predict what could happen for values that are not in the table.

Example 4

Write a function rule for the following table:

Number of CDs	2	4	6	8	10
Cost in \$	24	48	72	86	120

Solution

You pay \$24 for 2 CDs, \$48 for 4 CDs, \$120 for 10 CDs. That means that each CD costs \$12.

We can write a function rule:

Cost = \$12 \times (number of CDs) or $f(x) = 12x$

Example 5

Write a function rule for the following table:

x	-3	-2	-1	0	1	2	3
y	3	2	1	0	1	2	3

Solution

You can see that a negative number turns into the same number, only positive, while a non-negative number stays the same. This means that the function being used here is the absolute value function: $f(x) = |x|$.

Coming up with a function based on a set of values really is as tricky as it looks. There's no rule that will tell you the function every time, so you just have to think of all the types of functions you know and guess which one might be a good fit, and then check if your guess is right. In this book, though, we'll stick to writing functions for linear relationships, which are the simplest type of function.

Represent a Real-World Situation with a Function

Let's look at a few real-world situations that can be represented by a function.

Example 5

Maya has an internet service that currently has a monthly access fee of \$11.95 and a connection fee of \$0.50 per hour. Represent her monthly cost as a function of connection time.

Solution

Define

Let x = the number of hours Maya spends on the internet in one month

Let y = Maya's monthly cost

Translate

The cost has two parts: the one-time fee of \$11.95 and the per-hour charge of \$0.50. So the total cost is the flat fee + the charge per hour \times the number of hours.

Answer

The function is $y = f(x) = 11.95 + 0.50x$.

Example 6

Alfredo wants a deck build around his pool. The dimensions of the pool are 12 feet \times 24 feet and the decking costs \$3 per square foot. Write the cost of the deck as a function of the width of the deck.

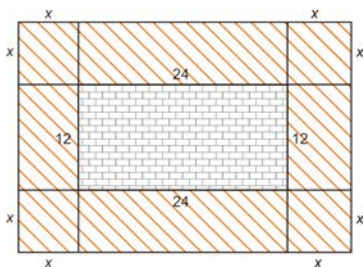
Solution

Define

Let x = width of the deck

Let y = cost of the deck

Make a sketch and label it



Translate

You can look at the decking as being formed by several rectangles and squares. We can find the areas of all the separate pieces and add them together:

$$\text{Area} = 12x + 12x + 24x + 24x + x^2 + x^2 + x^2 + x^2 = 72x + 4x^2$$

To find the total cost, we then multiply the area by the cost per square foot (\$3).

Answer

$$f(x) = 3(72x + 4x^2) = 216x + 12x^2$$

Example 7

A cell phone company sells two million phones in their first year of business. The number of phones they sell doubles each year. Write a function that gives the number of phones that are sold per year as a function of how old the company is.

Solution

Define

Let x = age of company in years

Let y = number of phones that are sold per year

Make a table

Age (years)	1	2	3	4	5	6	7
Millions of phones	2	4	8	16	32	64	128

Write a function rule

The number of phones sold per year doubles every year, so the first year the company sells 2 million phones, the next year it sells 2×2 million, the next year it sells $2 \times 2 \times 2$ million, and so on. You might remember that when we multiply a number by itself several times we can use exponential notation: $2 = 2^1$, $2 \times 2 = 2^2$, $2 \times 2 \times 2 = 2^3$, and so on. In this problem, the exponent just happens to match the company's age in years, which makes our function easy to describe.

Answer

$$y = f(x) = 2^x$$

Review Questions

- Identify the domain and range of the following functions.
 - Dustin charges \$10 per hour for mowing lawns.
 - Maria charges \$25 per hour for tutoring math, with a minimum charge of \$15.
 - $f(x) = 15x - 12$
 - $f(x) = 2x^2 + 5$
 - $f(x) = \frac{1}{x}$
 - $f(x) = \sqrt{x}$
- What is the range of the function $y = x^2 - 5$ when the domain is $-2, -1, 0, 1, 2$?
- What is the range of the function $y = 2x - \frac{3}{4}$ when the domain is $-2.5, -1.5, 5$?
- What is the domain of the function $y = 3x$ when the range is $9, 12, 15$?
- What is the range of the function $y = 3x$ when the domain is $9, 12, 15$?
- Angie makes \$6.50 per hour working as a cashier at the grocery store. Make a table that shows how much she earns if she works 5, 10, 15, 20, 25, or 30 hours.
- The area of a triangle is given by the formula $A = \frac{1}{2}bh$. If the base of the triangle measures 8 centimeters, make a table that shows the area of the triangle for heights 1, 2, 3, 4, 5, and 6 centimeters.
- Make a table of values for the function $f(x) = \sqrt{2}x + 3$ for input values $-1, 0, 1, 2, 3, 4, 5$.
- Write a function rule for the following table:

x	3	4	5	6
y	9	16	15	36

- Write a function rule for the following table:

Hours	0	1	2	3
Cost	15	20	25	30

11. Write a function rule for the following table:

x	0	1	2	3
y	24	12	6	3

12. Write a function that represents the number of cuts you need to cut a ribbon into x pieces.

13. Write a function that represents the number of cuts you need to divide a pizza into x slices.

14. Solomon charges a \$40 flat rate plus \$25 per hour to repair a leaky pipe.

(a) Write a function that represents the total fee charged as a function of hours worked.

(b) How much does Solomon earn for a 3-hour job?

(c) How much does he earn for three separate 1-hour jobs?

15. Rochelle has invested \$2500 in a jewelry making kit. She makes bracelets that she can sell for \$12.50 each.

(a) Write a function that shows how much money Rochelle makes from selling b bracelets.

(b) Write a function that shows how much money Rochelle has after selling b bracelets, minus her investment in the kit.

(c) How many bracelets does Rochelle need to make before she breaks even?

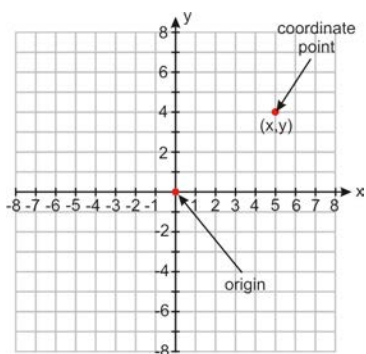
(d) If she buys a \$50 display case for her bracelets, how many bracelets does she now need to sell to break even?

1.6 Functions as Graphs

Learning Objectives

- Graph a function from a rule or table.
- Write a function rule from a graph.
- Analyze the graph of a real world situation.
- Determine whether a relation is a function.

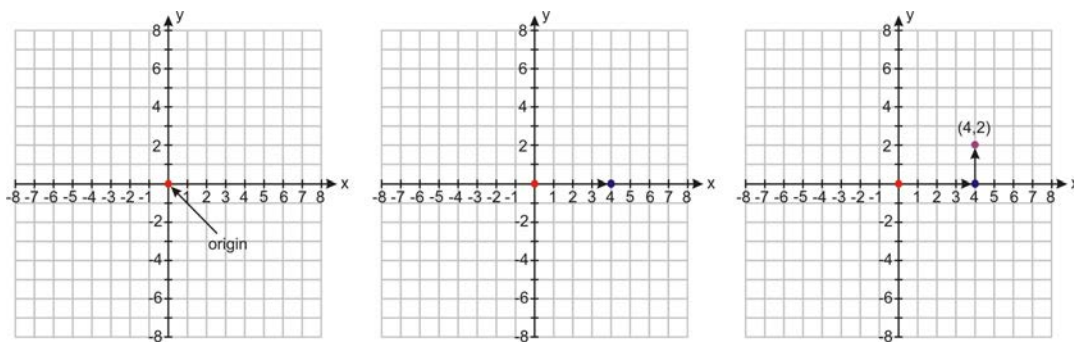
Introduction



We represent functions graphically by plotting points on a **coordinate plane** (also sometimes called the **Cartesian plane**). The coordinate plane is a grid formed by a horizontal number line and a vertical

number line that cross at a point called the **origin**. The origin has this name because it is the “starting” location; every other point on the grid is described in terms of how far it is from the origin.

The horizontal number line is called the x -**axis** and the vertical line is called the y -**axis**. We can represent each value of a function as a point on the plane by representing the x -value as a distance along the x -axis and the y -value as a distance along the y -axis. For example, if the y -value of a function is 2 when the x -value is 4, we can represent this pair of values with a point that is 4 units to the right of the origin (that is, 4 units along the x -axis) and 2 units up (2 units in the y -direction).



We write the location of this point as $(4, 2)$.

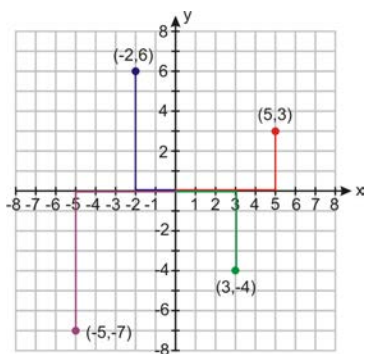
Example 1

Plot the following coordinate points on the Cartesian plane.

- a) $(5, 3)$
- b) $(-2, 6)$
- c) $(3, -4)$
- d) $(-5, -7)$

Solution

Here are all the coordinate points on the same plot.



Notice that we move to the right for a positive x -value and to the left for a negative one, just as we would on a single number line. Similarly, we move up for a positive y -value and down for a negative one.

The x - and y -axes divide the coordinate plane into four **quadrants**. The quadrants are numbered counter-clockwise starting from the upper right, so the plotted point for (a) is in the **first** quadrant, (b) is in the **second** quadrant, (c) is in the **fourth** quadrant, and (d) is in the **third** quadrant.

Graph a Function From a Rule or Table

If we know a rule or have a table of values that describes a function, we can draw a graph of the function. A table of values gives us coordinate points that we can plot on the Cartesian plane.

Example 2

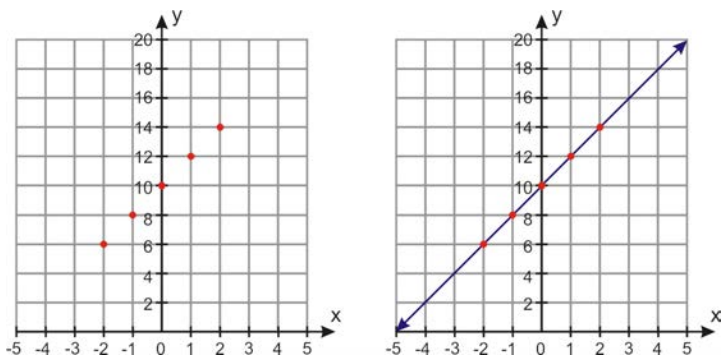
Graph the function that has the following table of values.

x	-2	-1	0	1	2
y	6	8	10	12	14

Solution

The table gives us five sets of coordinate points: $(-2, 6)$, $(-1, 8)$, $(0, 10)$, $(1, 12)$, and $(2, 14)$.

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function or given a real-world context, we can just assume that the domain is the set of all real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth line (which, we understand, continues infinitely in both directions).



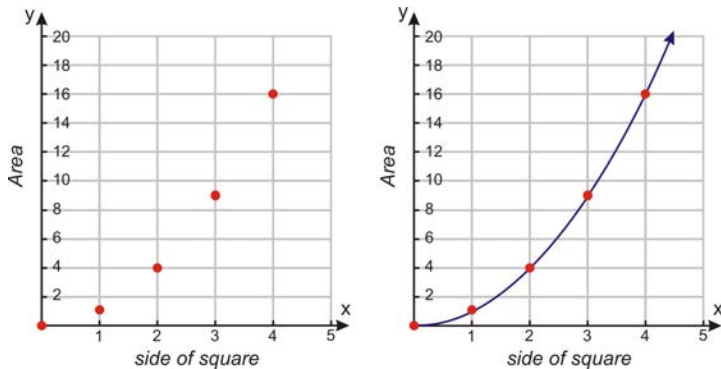
Example 3

Graph the function that has the following table of values.

Side of square	0	1	2	3	4
Area of square	0	1	4	9	16

The table gives us five sets of coordinate points: $(0, 0)$, $(1, 1)$, $(2, 4)$, $(3, 9)$, and $(4, 16)$.

To graph the function, we plot all the coordinate points. Since we are not told the domain of the function, we can assume that the domain is the set of all non-negative real numbers. To show that the function holds for all values in the domain, we connect the points with a smooth curve. The curve does not make sense for negative values of the independent variable, so it stops at $x = 0$, but it continues infinitely in the positive direction.



Example 4

Graph the function that has the following table of values.

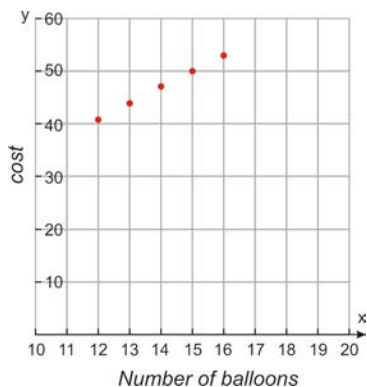
Number of balloons	12	13	14	15	16
Cost	41	44	47	50	53

This function represents the total cost of the balloons delivered to your house. Each balloon is \$3 and the store delivers if you buy a dozen balloons or more. The delivery charge is a \$5 flat fee.

Solution

The table gives us five sets of coordinate points: (12, 41), (13, 44), (14, 47), (15, 50), and (16, 53).

To graph the function, we plot all the coordinate points. Since the x -values represent the number of balloons for 12 balloons or more, the domain of this function is all integers greater than or equal to 12. In this problem, the points are not connected by a line or curve because it doesn't make sense to have non-integer values of balloons.



In order to draw a graph of a function given the function rule, we must first make a table of values to give us a set of points to plot. Choosing good values for the table is a skill you'll develop throughout this course. When you pick values, here are some of the things you should keep in mind.

- Pick only values from the domain of the function.
- If the domain is the set of real numbers or a subset of the real numbers, the graph will be a continuous curve.
- If the domain is the set of integers or a subset of the integers, the graph will be a set of points not connected by a curve.
- Picking integer values is best because it makes calculations easier, but sometimes we need to pick other values to capture all the details of the function.

- Often we start with one set of values. Then after drawing the graph, we realize that we need to pick different values and redraw the graph.

Example 5

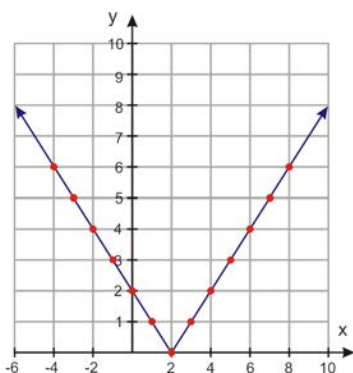
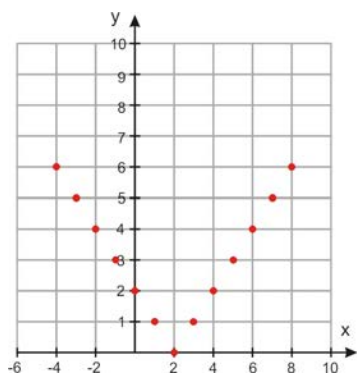
Graph the following function: $f(x) = |x - 2|$

Solution

Make a table of values. Pick a variety of negative and positive values for x . Use the function rule to find the value of y for each value of x . Then, graph each of the coordinate points.

Table 1.8:

x	$y = f(x) = x - 2 $
-4	$ -4 - 2 = -6 = 6$
-3	$ -3 - 2 = -5 = 5$
-2	$ -2 - 2 = -4 = 4$
-1	$ -1 - 2 = -3 = 3$
0	$ 0 - 2 = -2 = 2$
1	$ 1 - 2 = -1 = 1$
2	$ 2 - 2 = 0 = 0$
3	$ 3 - 2 = 1 = 1$
4	$ 4 - 2 = 2 = 2$
5	$ 5 - 2 = 3 = 3$
6	$ 6 - 2 = 4 = 4$
7	$ 7 - 2 = 5 = 5$
8	$ 8 - 2 = 6 = 6$



It is wise to work with a lot of values when you begin graphing. As you learn about different types of functions, you will start to only need a few points in the table of values to create an accurate graph.

Example 6

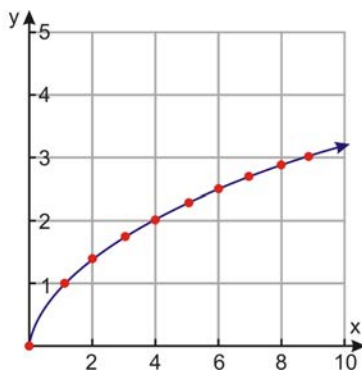
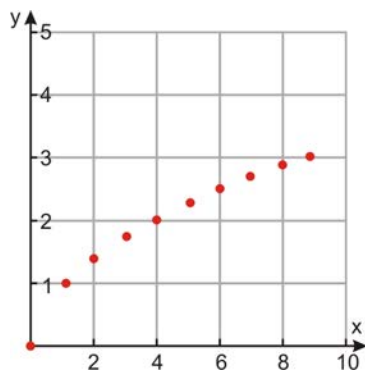
Graph the following function: $f(x) = \sqrt{x}$

Solution

Make a table of values. We know x can't be negative because we can't take the square root of a negative number. The domain is all positive real numbers, so we pick a variety of positive integer values for x . Use the function rule to find the value of y for each value of x .

Table 1.9:

x	$y = f(x) = \sqrt{x}$
0	$\sqrt{0} = 0$
1	$\sqrt{1} = 1$
2	$\sqrt{2} \approx 1.41$
3	$\sqrt{3} \approx 1.73$
4	$\sqrt{4} = 2$
5	$\sqrt{5} \approx 2.24$
6	$\sqrt{6} \approx 2.45$
7	$\sqrt{7} \approx 2.65$
8	$\sqrt{8} \approx 2.83$
9	$\sqrt{9} = 3$



Note that the range is all positive real numbers.

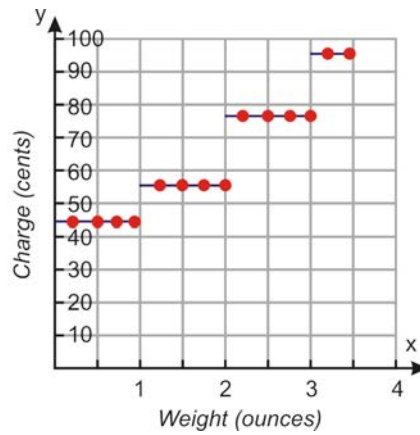
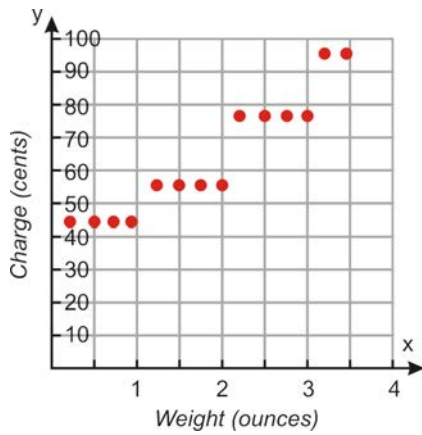
Example 7

The post office charges 41 cents to send a letter that is one ounce or less and an extra 17 cents for each additional ounce or fraction of an ounce. This rate applies to letters up to 3.5 ounces.

Solution

Make a table of values. We can't use negative numbers for x because it doesn't make sense to have negative weight. We pick a variety of positive values for x , making sure to include some decimal values because prices can be decimals too. Then we use the function rule to find the value of y for each value of x .

x	0	0.2	0.5	0.8	1	1.2	1.5	1.8	2	2.2	2.5	2.8	3	3.2	3.5
y	0	41	41	41	41	58	58	58	58	75	75	75	75	92	92

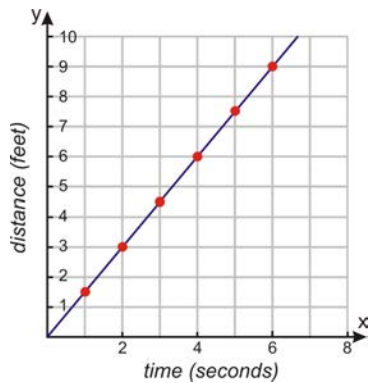


Write a Function Rule from a Graph

Sometimes you'll need to find the equation or rule of a function by looking at the graph of the function. Points that are on the graph can give you values of dependent and independent variables that are related to each other by the function rule. However, you must make sure that the rule works for all the points on the curve. In this course you will learn to recognize different kinds of functions and discover the rules for all of them. For now we'll look at some simple examples and find patterns that will help us figure out how the dependent and independent variables are related.

Example 8

The graph to the right shows the distance that an ant covers over time. Find the function rule that shows how distance and time are related to each other.



Solution

Let's make a table of values of several coordinate points to see if we can spot how they are related to each other.

Time	0	1	2	3	4	5	6
Distance	0	1.5	3	4.5	6	7.5	9

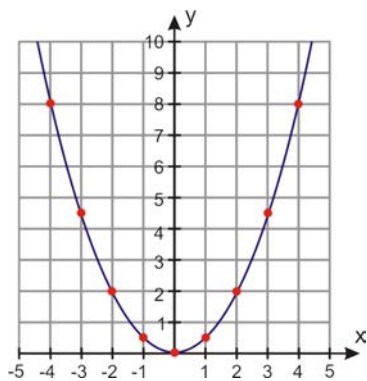
We can see that for every second the distance increases by 1.5 feet. We can write the function rule as

$$\text{Distance} = 1.5 \times \text{time}$$

The equation of the function is $f(x) = 1.5x$.

Example 9

Find the function rule that describes the function shown in the graph.



Solution

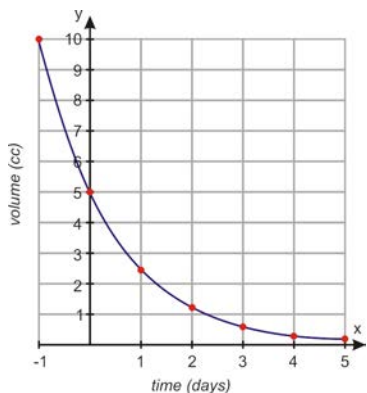
Again, we can make a table of values of several coordinate points to identify how they are related to each other.

x	-4	-3	-2	-1	0	1	2	3	4
y	8	4.5	2	.5	0	.5	5	4.5	8

Notice that the values of y are half of perfect squares: 8 is half of 16 (which is 4 squared), 4.5 is half of 9 (which is 3 squared), and so on. So the equation of the function is $f(x) = \frac{1}{2}x^2$.

Example 10

Find the function rule that shows the volume of a balloon at different times, based on the following graph:



(Notice that the graph shows negative time. The negative time can represent what happened on days before you started measuring the volume.)

Solution

Once again, we make a table to spot the pattern:

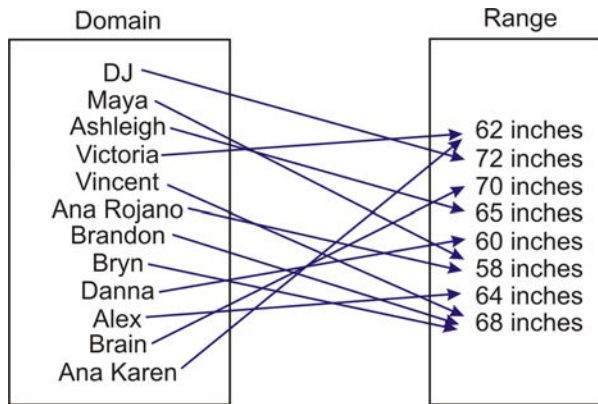
Time	-1	0	1	2	3	4	5
Volume	10	5	2.5	1.2	0.6	0.3	0.15

We can see that every day, the volume of the balloon is half what it was the previous day. On day 0, the volume is 5; on day 1, the volume is $5 \times \frac{1}{2}$; on day 2, it is $5 \times \frac{1}{2} \times \frac{1}{2}$, and in general, on day x it is $5 \times \left(\frac{1}{2}\right)^x$. The equation of the function is $f(x) = 5 \times \left(\frac{1}{2}\right)^x$.

Determine Whether a Relation is a Function

A function is a special kind of **relation**. In a function, for each input there is exactly one output; in a relation, there can be more than one output for a given input.

Consider the relation that shows the heights of all students in a class. The domain is the set of people in the class and the range is the set of heights. This relation is a function because each person has exactly one height. If any person had more than one height, the relation would not be a function.



Notice that even though the same person can't have more than one height, it's okay for more than one person to have the same height. In a function, more than one input can have the same output, as long as more than one output never comes from the same input.

Example 11

Determine if the relation is a function.

- a) $(1, 3), (-1, -2), (3, 5), (2, 5), (3, 4)$
- b) $(-3, 20), (-5, 25), (-1, 5), (7, 12), (9, 2)$
- c)

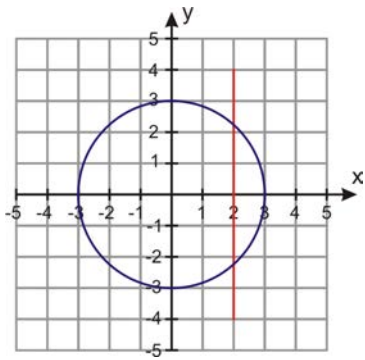
x	2	1	0	1	2
y	12	10	8	6	4

Solution

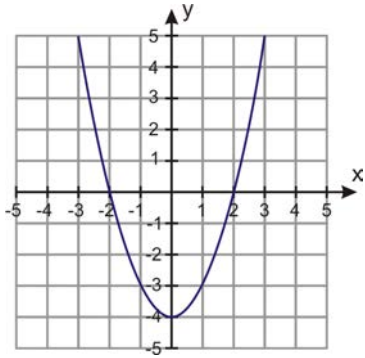
The easiest way to figure out if a relation is a function is to look at all the x -values in the list or the table. If a value of x appears more than once, and it's paired up with different y -values, then the relation is not a function.

- a) You can see that in this relation there are two different y -values paired with the x -value of 3. This means that this relation is **not** a function.
- b) Each value of x has exactly one y -value. The relation is a function.
- c) In this relation there are two different y -values paired with the x -value of 2 and two y -values paired with the x -value of 1. The relation is **not** a function.

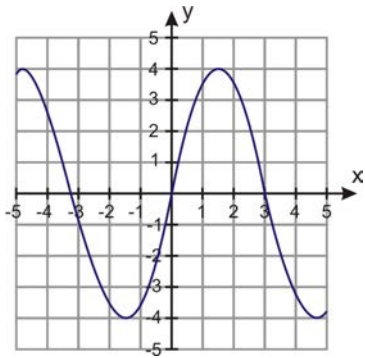
When a relation is represented graphically, we can determine if it is a function by using the **vertical line test**. If you can draw a vertical line that crosses the graph in more than one place, then the relation is not a function. Here are some examples.



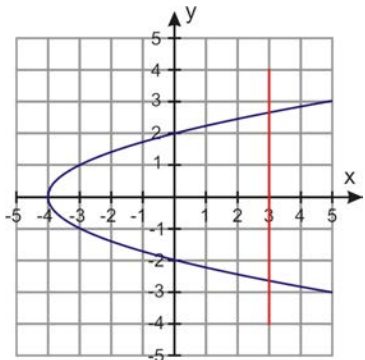
Not a function. It fails the vertical line test.



A function. No vertical line will cross more than one point on the graph.



A function. No vertical line will cross more than one point on the graph.



Not a function. It fails the vertical line test.

Additional Resources

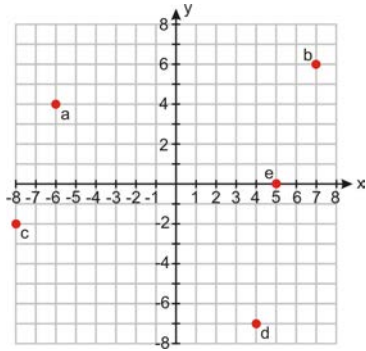
Once you've had some practice graphing functions by hand, you may want to use a graphing calculator to make graphing easier. If you don't have one, you can also use the applet at <http://rechmeronline.de/function-graphs/>. Just type a function in the blank and press Enter. You can use the options under Display Properties to zoom in or pan around to different parts of the graph.

Review Questions

1. Plot the coordinate points on the Cartesian plane.

- (a) (4, -4)
- (b) (2, 7)
- (c) (-3, -5)
- (d) (6, 3)
- (e) (-4, 3)

2. Give the coordinates for each point in this Cartesian plane.



3. Graph the function that has the following table of values.

(a)

x	-10	-5	0	5	10
y	-3	-0.5	2	4.5	7

(b)

Side of cube (in.)	0	1	2	3
Volume (in ³)	0	1	8	27

(c)

Time (hours)	-2	-1	0	1	2
Distance from town center (miles)	50	25	0	25	50

4. Graph the following functions.

- (a) Brandon is a member of a movie club. He pays a \$50 annual membership and \$8 per movie.
- (b) $f(x) = (x - 2)^2$
- (c) $f(x) = 3 \cdot 2^x$

5. Determine whether each relation is a function:

(a) (1, 7), (2, 7), (3, 8), (4, 8), (5, 9)

(b) (1, 1), (1, -1), (4, 2), (4, -2), (9, 3), (9, -3)

(c)

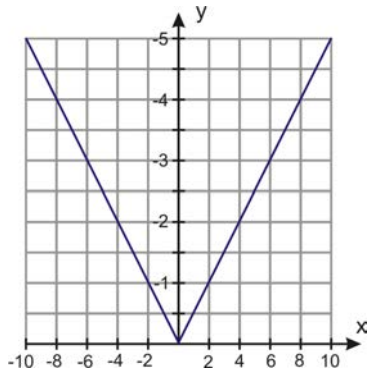
x	-4	-3	-2	-1	0
y	16	9	4	1	0

(d)

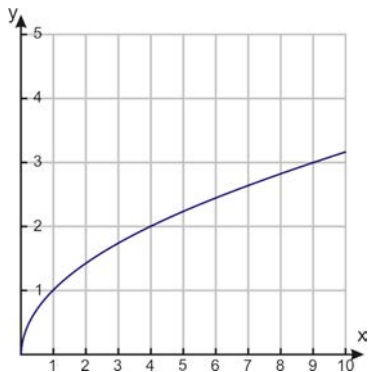
Age	20	25	25	30	35
Number of jobs by that age	3	4	7	4	2

6. Write the function rule for each graph.

(a)

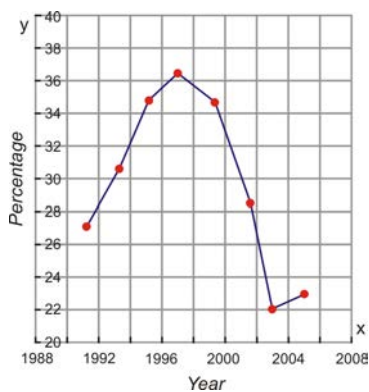


(b)



7. The students at a local high school took The Youth Risk Behavior Survey. The graph below shows the percentage of high school students who reported that they were current smokers. (A current smoker is anyone who has smoked one or more cigarettes in the past 30 days.) What percentage of high-school students were current smokers in the following years?

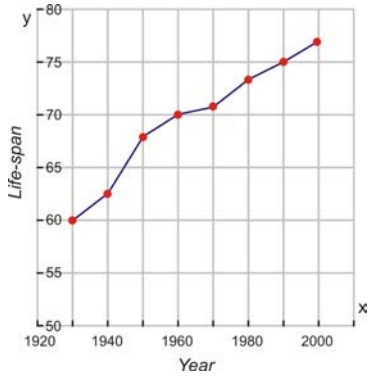
- (a) 1991
- (b) 1996
- (c) 2004
- (d) 2005



8. The graph below shows the average life-span of people based on the year in which they were born. This information comes from the National Vital Statistics Report from the Center for Disease Control. What is the average life-span of a person born in the following years?

- (a) 1940
- (b) 1955
- (c) 1980

(d) 1995

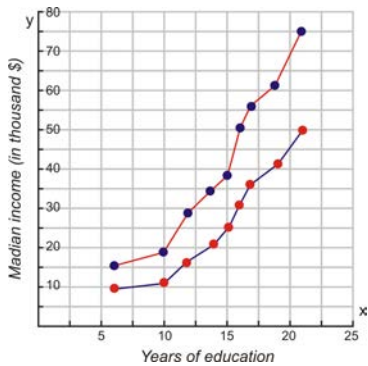


9. The graph below shows the median income of an individual based on his/her number of years of education. The top curve shows the median income for males and the bottom curve shows the median income for females. (Source: US Census, 2003.) What is the median income of a male that has the following years of education?

- (a) 10 years of education
- (b) 17 years of education

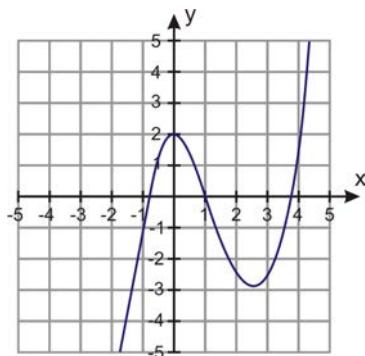
10. What is the median income of a female that has the same years of education?

- (a) 10 years of education
- (b) 17 years of education

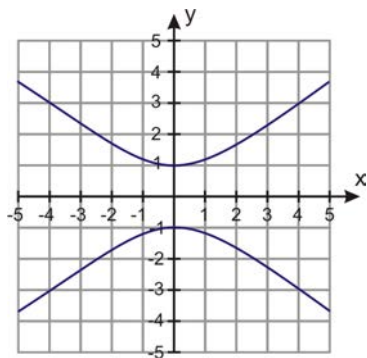


11. Use the vertical line test to determine whether each relation is a function.

(a)



(b)



1.7 Problem-Solving Plan

Learning Objectives

- Read and understand given problem situations.
- Make a plan to solve the problem.
- Solve the problem and check the results.
- Compare alternative approaches to solving the problem.
- Solve real-world problems using a plan.

Introduction

We always think of mathematics as the subject in school where we solve lots of problems. Problem solving is necessary in all aspects of life. Buying a house, renting a car, or figuring out which is the better sale are just a few examples of situations where people use problem-solving techniques. In this book, you will learn different strategies and approaches to solving problems. In this section, we will introduce a problem-solving plan that will be useful throughout this book.

Read and Understand a Given Problem Situation

The first step to solving a word problem is to **read and understand** the problem. Here are a few questions that you should be asking yourself:

- What am I trying to find out?
- What information have I been given?
- Have I ever solved a similar problem?

This is also a good time to define any variables. When you identify your **knowns** and **unknowns**, it is often useful to assign them a letter to make notation and calculations easier.

Make a Plan to Solve the Problem

The next step in the problem-solving plan is to **develop a strategy**. How can the information you know assist you in figuring out the unknowns?

Here are some common strategies that you will learn:

- Drawing a diagram.
- Making a table.
- Looking for a pattern.
- Using guess and check.
- Working backwards.
- Using a formula.
- Reading and making graphs.
- Writing equations.
- Using linear models.
- Using dimensional analysis.
- Using the right type of function for the situation.

In most problems, you will use a combination of strategies. For example, looking for patterns is a good strategy for most problems, and making a table and drawing a graph are often used together. The “writing an equation” strategy is the one you will work with the most in your study of algebra.

Solve the Problem and Check the Results

Once you develop a plan, you can implement it and **solve the problem**, carrying out all operations to arrive at the answer you are seeking.

The last step in solving any problem should always be to **check and interpret** the answer. Ask yourself:

- Does the answer make sense?
- If you plug the answer back into the problem, do all the numbers work out?
- Can you get the same answer through another method?

Compare Alternative Approaches to Solving the Problem

Sometimes one specific method is best for solving a problem. Most problems, however, can be solved by using several different strategies. When you are familiar with all of the problem-solving strategies, it is up to you to choose the methods that you are most comfortable with and that make sense to you. In this book, we will often use more than one method to solve a problem, so we can demonstrate the strengths and weakness of different strategies for solving different types of problems.

Whichever strategy you are using, you should always implement the problem-solving plan when you are solving word problems. Here is a summary of the problem-solving plan.

Step 1:

Understand the problem

Read the problem carefully. Once the problem is read, list all the components and data that are involved. This is where you will be assigning your variables.

Step 2:

Devise a plan - Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table as a start to solve your problem solving plan.

Step 3:

Carry out the plan - Solve

This is where you solve the equation you developed in Step 2.

Step 4:

Look - Check and Interpret

Check to see if you used all your information. Then look to see if the answer makes sense.

Solve Real-World Problems Using a Plan

Let's now apply this problem solving plan to a problem.

Example 1

*A coffee maker is on sale at 50% off the regular ticket price. On the "Sunday Super Sale" the same coffee maker is on sale at an **additional** 40% off. If the final price is \$21, what was the original price of the coffee maker?*

Solution

Step 1: Understand

We know: A coffee maker is discounted 50% and then 40%. The final price is \$21.

We want: The original price of the coffee maker.

Step 2: Strategy

Let's look at the given information and try to find the relationship between the information we know and the information we are trying to find.

50% off the original price means that the sale price is half of the original or $0.5 \times$ original price.

So, the first sale price = $0.5 \times$ original price

A savings of 40% off the new price means you pay 60% of the new price, or $0.6 \times$ new price.

$0.6 \times (0.5 \times \text{original price}) = 0.3 \times \text{original price}$ is the price after the second discount.

We know that after two discounts, the final price is \$21.

So $0.3 \times \text{original price} = \21 .

Step 3: Solve

Since $0.3 \times \text{original price} = \21 , we can find the original price by dividing \$21 by 0.3.

Original price = $\$21 \div 0.3 = \70 .

The original price of the coffee maker was \$70.

Step 4: Check

We found that the original price of the coffee maker is \$70.

To check that this is correct, let's apply the discounts.

50% of \$70 = $.5 \times \$70 = \35 savings. So the price after the first discount is original price – savings or $\$70 - 35 = \35 .

Then 40% of that is $.4 \times \$35 = \14 . So after the second discount, the price is $\$35 - 14 = \21 .

The answer checks out.

Additional Resources

The problem-solving plan used here is based on the ideas of George Pólya, who describes his useful problem-solving strategies in more detail in the book *How to Solve It*. Some of the techniques in the book can also be found on Wikipedia, in the entry http://en.wikipedia.org/wiki/How_to_Solve_It.

Review Questions

1. A sweatshirt costs \$35. Find the total cost if the sales tax is 7.75%.
2. This year you got a 5% raise. If your new salary is \$45,000, what was your salary before the raise?
3. Mariana deposits \$500 in a savings account that pays 3% simple interest per year. How much will be in her account after three years?
4. It costs \$250 to carpet a room that is 14 ft by 18 ft. How much does it cost to carpet a room that is 9 ft by 10 ft?
5. A department store has a 15% discount for employees. Suppose an employee has a coupon worth \$10 off any item and she wants to buy a \$65 purse. What is the final cost of the purse if the employee discount is applied before the coupon is subtracted?
6. To host a dance at a hotel you must pay \$250 plus \$20 per guest. How much money would you have to pay for 25 guests?
7. Yusef's phone plan costs \$10 a month plus \$0.05 per minute. If his phone bill for last month was \$25.80, how many minutes did he spend on the phone?
8. It costs \$12 to get into the San Diego County Fair and \$1.50 per ride.
 - (a) If Rena spent \$24 in total, how many rides did she go on?
 - (b) How much would she have spent in total if she had gone on five more rides?
9. An ice cream shop sells a small cone for \$2.95, a medium cone for \$3.50, and a large cone for \$4.25. Last Saturday, the shop sold 22 small cones, 26 medium cones and 15 large cones. How much money did the store earn?
10. In Lise's chemistry class, there are two midterm exams, each worth 30% of her total grade, and a final exam worth 40%. If Lise scores 90% on both midterms and 80% on the final exam, what is her overall score in the class?
11. The sum of the angles in a triangle is 180 degrees. If the second angle is twice the size of the first angle and the third angle is three times the size of the first angle, what are the measures of the angles in the triangle?
12. A television that normally costs \$120 goes on sale for 20% off. What is the new price?
13. A cake recipe calls for $1\frac{3}{4}$ cup of flour. Jeremy wants to make four cakes. How many cups of flour will he need?
14. Casey is twice as old as Marietta, who is two years younger than Jake. If Casey is 14, how old is Jake?
15. Kylie is mowing lawns to earn money for a new bike. After mowing four lawns, she still needs \$40 more to pay for the bike. After mowing three more lawns, she has \$5 more than she needs to pay for the bike.
 - (a) How much does she earn per lawn?
 - (b) What is the cost of the bike?
16. Jared goes trick-or-treating with his brother and sister. At the first house they stop at, they collect three pieces of candy each; at the next three houses, they collect two pieces of candy each. Then they split up and go down different blocks, where Jared collects 12 pieces of candy and his brother and sister collect 14 each.
 - (a) How many pieces of candy does Jared end up with?

- (b) How many pieces of candy do all three of them together end up with?
17. Marco's daughter Elena has four boxes of toy blocks, with 50 blocks in each one. One day she dumps them all out on the floor, and some of them get lost. When Marco tries to put them away again, he ends up with 45 blocks in one box, 53 in another, 46 in a third, and 51 in the fourth. How many blocks are missing?
 18. A certain hour-long TV show usually includes 16 minutes of commercials. If the season finale is two and a half hours long, how many minutes of commercials should it include to keep the same ratio of commercial time to show time?
 19. Karen and Chase bet on a baseball game: if the home team wins, Karen owes Chase fifty cents for every run scored by both teams, and Chase owes Karen the same amount if the visiting team wins. The game runs nine innings, and the home team scores one run in every odd-numbered inning, while the visiting team scores two runs in the third inning and two in the sixth. Who owes whom how much?
 20. Kelly, Chris, and Morgan are playing a card game. In this game, the first player to empty their hand scores points for all the cards left in the other players' hands as follows: aces are worth one point, face cards ten points, and all other cards are face value. When Kelly empties her hand, Morgan is holding two aces, a king, and a three; Chris is holding a five, a seven, and a queen. How many points does Kelly score?
 21. A local club rents out a social hall to host an event. The hall rents for \$350, and they hope to make back the rental price by charging \$15 admission per person. How many people need to attend for the club to break even?
 22. You plan to host a barbecue, and you expect 10 friends, 8 neighbors, and 7 relatives to show up.
 - (a) If you expect each person (including yourself) to eat about two ounces of potato salad, how many half-pound containers of potato salad should you buy?
 - (b) If hot dogs come in ten-packs that cost \$4.80 apiece and hot dog buns come in eight-packs that cost \$2.80 apiece, how much will you need to spend to have hot dogs and buns for everyone?

1.8 Problem-Solving Strategies: Make a Table and Look for a Pattern

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy “make a table.”
- Develop and use the strategy “look for a pattern.”
- Plan and compare alternative approaches to solving a problem.
- Solve real-world problems using the above strategies as part of a plan.

Introduction

In this section, we will apply the problem-solving plan you learned about in the last section to solve several real-world problems. You will learn how to develop and use the methods **make a table** and **look for a pattern**.

Read and Understand Given Problem Situations

The most difficult parts of problem-solving are most often the first two steps in our problem-solving plan. You need to read the problem and make sure you understand what you are being asked. Once you understand the problem, you can devise a strategy to solve it.

Let's apply the first two steps to the following problem.

Example 1:

Six friends are buying pizza together and they are planning to split the check equally. After the pizza was ordered, one of the friends had to leave suddenly, before the pizza arrived. Everyone left had to pay \$1 extra as a result. How much was the total bill?

Solution

Understand

We want to find how much the pizza cost.

We know that five people had to pay an extra \$1 each when one of the original six friends had to leave.

Strategy

We can start by making a list of possible amounts for the total bill.

We divide the amount by six and then by five. The total divided by five should equal \$1 more than the total divided by six.

Look for any patterns in the numbers that might lead you to the correct answer.

In the rest of this section you will learn how to make a table or look for a pattern to figure out a solution for this type of problem. After you finish reading the rest of the section, you can finish solving this problem for homework.

Develop and Use the Strategy: Make a Table

The method "Make a Table" is helpful when solving problems involving numerical relationships. When data is organized in a table, it is easier to recognize patterns and relationships between numbers. Let's apply this strategy to the following example.

Example 2

Josie takes up jogging. On the first week she jogs for 10 minutes per day, on the second week she jogs for 12 minutes per day. Each week, she wants to increase her jogging time by 2 minutes per day. If she jogs six days each week, what will be her total jogging time on the sixth week?

Solution

Understand

We know in the first week Josie jogs 10 minutes per day for six days.

We know in the second week Josie jogs 12 minutes per day for six days.

Each week, she increases her jogging time by 2 minutes per day and she jogs 6 days per week.

We want to find her total jogging time in week six.

Strategy

A good strategy is to list the data we have been given in a table and use the information we have been given to find new information.

We are told that Josie jogs 10 minutes per day for six days in the first week and 12 minutes per day for six days in the second week. We can enter this information in a table:

Table 1.10:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72

You are told that each week Josie increases her jogging time by 2 minutes per day and jogs 6 times per week. We can use this information to continue filling in the table until we get to week six.

Table 1.11:

Week	Minutes per Day	Minutes per Week
1	10	60
2	12	72
3	14	84
4	16	96
5	18	108
6	20	120

Apply strategy/solve

To get the answer we read the entry for week six.

Answer: In week six Josie jogs a total of 120 minutes.

Check

Josie increases her jogging time by two minutes per day. She jogs six days per week. This means that she increases her jogging time by 12 minutes per week.

Josie starts at 60 minutes per week and she increases by 12 minutes per week for five weeks.

That means the total jogging time is $60 + 12 \times 5 = 120$ *minutes*.

The answer checks out.

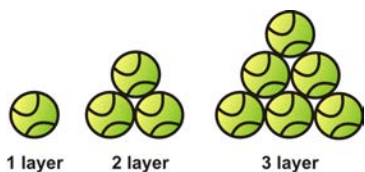
You can see that making a table helped us organize and clarify the information we were given, and helped guide us in the next steps of the problem. We solved this problem solely by making a table; in many situations, we would combine this strategy with others to get a solution.

Develop and Use the Strategy: Look for a Pattern

Looking for a pattern is another strategy that you can use to solve problems. The goal is to look for items or numbers that are repeated or a series of events that repeat. The following problem can be solved by finding a pattern.

Example 3

You arrange tennis balls in triangular shapes as shown. How many balls will there be in a triangle that has 8 rows?



Solution

Understand

We know that we arrange tennis balls in triangles as shown.

We want to know how many balls there are in a triangle that has 8 rows.

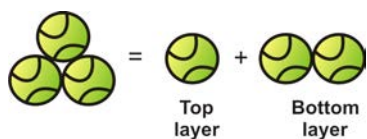
Strategy

A good strategy is to make a table and list how many balls are in triangles of different rows.

One row: It is simple to see that a triangle with one row has only one ball.



Two rows: For a triangle with two rows, we add the balls from the top row to the balls from the bottom row. It is useful to make a sketch of the separate rows in the triangle.



$$3 = 1 + 2$$

Three rows: We add the balls from the top triangle to the balls from the bottom row.



$$6 = 3 + 3$$

Now we can fill in the first three rows of a table.

Table 1.12:

Number of Rows	Number of Balls
1	1
2	3
3	6

We can see a pattern.

To create the next triangle, we add a new bottom row to the existing triangle.

The new bottom row has the same number of balls as there are rows. (For example, a triangle with 3 rows has 3 balls in the bottom row.)

To get the total number of balls for the new triangle, we add the number of balls in the old triangle to the number of balls in the new bottom row.

Apply strategy/solve:

We can complete the table by following the pattern we discovered.

Number of balls = number of balls in previous triangle + number of rows in the new triangle

Table 1.13:

Number of Rows	Number of Balls
1	1
2	3
3	6
4	$6 + 4 = 10$
5	$10 + 5 = 15$
6	$15 + 6 = 21$
7	$21 + 7 = 28$
8	$28 + 8 = 36$

Answer There are 36 balls in a triangle arrangement with 8 rows.

Check

Each row of the triangle has one more ball than the previous one. In a triangle with 8 rows, row 1 has 1 ball, row 2 has 2 balls, row 3 has 3 balls, row 4 has 4 balls, row 5 has 5 balls, row 6 has 6 balls, row 7 has 7 balls, row 8 has 8 balls.

When we add these we get: $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ balls

The answer checks out.

Notice that in this example we made tables and drew diagrams to help us organize our information and find a pattern. Using several methods together is a very common practice and is very useful in solving word problems.

Plan and Compare Alternative Approaches to Solving Problems

In this section, we will compare the methods of “Making a Table” and “Looking for a Pattern” by using each method in turn to solve a problem.

Example 4

Andrew cashes a \$180 check and wants the money in \$10 and \$20 bills. The bank teller gives him 12 bills. How many of each kind of bill does he receive?

Solution

Method 1: Making a Table

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table of the different ways Andrew can have twelve bills in tens and twenties. Andrew could have twelve \$10 bills and zero \$20 bills, or eleven \$10 bills and one \$20 bill, and so on. We can calculate the total amount of money for each case.

Apply strategy/solve

Table 1.14:

\$10 bills	\$ 20 bills	Total amount
12	0	$\$10(12) + \$20(0) = \$120$
11	1	$\$10(11) + \$20(1) = \$130$
10	2	$\$10(10) + \$20(2) = \$140$
9	3	$\$10(9) + \$20(3) = \$150$
8	4	$\$10(8) + \$20(4) = \$160$
7	5	$\$10(7) + \$20(5) = \$170$
6	6	$\$10(6) + \$20(6) = \$180$
5	7	$\$10(5) + \$20(7) = \$190$
4	8	$\$10(4) + \$20(8) = \$200$
3	9	$\$10(3) + \$20(9) = \$210$
2	10	$\$10(2) + \$20(10) = \$220$
1	11	$\$10(1) + \$20(11) = \$230$
0	12	$\$10(0) + \$20(12) = \$240$

In the table we listed all the possible ways you can get twelve \$10 bills and \$20 bills and the total amount of money for each possibility. The correct amount is given when Andrew has six \$10 bills and six \$20 bills.

Answer: Andrew gets six \$10 bills and six \$20 bills.

Check

Six \$10 bills and six \$20 bills $\rightarrow 6(\$10) + 6(\$20) = \$60 + \$120 = \$180$

The answer checks out.

Let's solve the same problem using the method "Look for a Pattern."

Method 2: Looking for a Pattern

Understand

Andrew gives the bank teller a \$180 check.

The bank teller gives Andrew 12 bills. These bills are a mix of \$10 bills and \$20 bills.

We want to know how many of each kind of bill Andrew receives.

Strategy

Let's start by making a table just as we did above. However, this time we will look for patterns in the table that can be used to find the solution.

Apply strategy/solve

Let's fill in the rows of the table until we see a pattern.

Table 1.15:

\$10 bills	\$20 bills	Total amount
12	0	$\$10(12) + \$20(0) = \$120$
11	1	$\$10(11) + \$20(1) = \$130$
10	2	$\$10(10) + \$20(2) = \$140$

We see that every time we reduce the number of \$10 bills by one and increase the number of \$20 bills by one, the total amount increases by \$10. The last entry in the table gives a total amount of \$140, so we have \$40 to go until we reach our goal. This means that we should reduce the number of \$10 bills by four and increase the number of \$20 bills by four. That would give us six \$10 bills and six \$20 bills.

$$6(\$10) + 6(\$20) = \$60 + 120 = \$180$$

Answer: Andrew gets six \$10 bills and six \$20 bills.

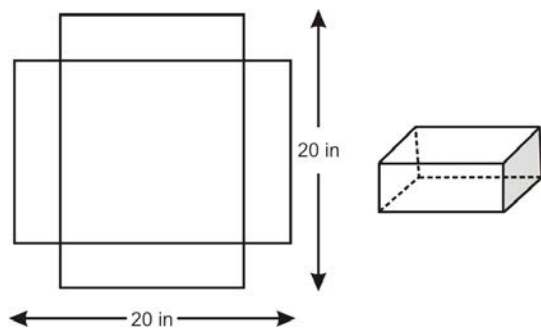
Check

Six \$10 bills and six \$20 bills $\rightarrow 6(\$10) + 6(\$20) = \$60 + 120 = \180

The answer checks out.

You can see that the second method we used for solving the problem was less tedious. In the first method, we listed all the possible options and found the answer we were seeking. In the second method, we started by listing the options, but we found a pattern that helped us find the solution faster. The methods of “Making a Table” and “Looking for a Pattern” are both more powerful if used alongside other problem-solving methods.

Solve Real-World Problems Using Selected Strategies as Part of a Plan



Example 5

Anne is making a box without a lid. She starts with a 20 in. square piece of cardboard and cuts out four equal squares from each corner of the cardboard as shown. She then folds the sides of the box and glues the edges together. How big does she need to cut the corner squares in order to make the box with the biggest volume?

Solution

Step 1:

Understand

Anne makes a box out of a 20 in × 20 in piece of cardboard.

She cuts out four equal squares from the corners of the cardboard.

She folds the sides and glues them to make a box.

How big should the cut out squares be to make the box with the biggest volume?

Step 2:

Strategy

We need to remember the formula for the volume of a box.

$$\text{Volume} = \text{Area of base} \times \text{height}$$

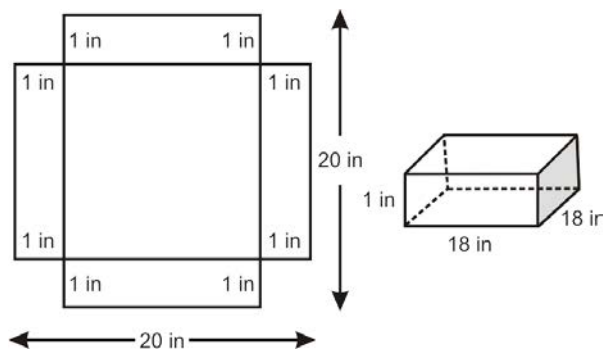
$$\text{Volume} = \text{width} \times \text{length} \times \text{height}$$

Make a table of values by picking different values for the side of the squares that we are cutting out and calculate the volume.

Step 3:

Apply strategy/solve

Let's "make" a box by cutting out four corner squares with sides equal to 1 inch. The diagram will look like this:



You see that when we fold the sides over to make the box, the height becomes 1 inch, the width becomes 18 inches and the length becomes 18 inches.

$$\text{Volume} = \text{width} \times \text{length} \times \text{height}$$

$$\text{Volume} = 18 \times 18 \times 1 = 324 \text{ in}^3$$

Let's make a table that shows the value of the box for different square sizes:

Table 1.16:

Side of Square	Box Height	Box Width	Box Length	Volume
1	1	18	18	$18 \times 18 \times 1 = 324$
2	2	16	16	$16 \times 16 \times 2 = 512$
3	3	14	14	$14 \times 14 \times 3 = 588$
4	4	12	12	$12 \times 12 \times 4 = 576$
5	5	10	10	$10 \times 10 \times 5 = 500$
6	6	8	8	$8 \times 8 \times 6 = 384$
7	7	6	6	$6 \times 6 \times 7 = 252$
8	8	4	4	$4 \times 4 \times 8 = 128$
9	9	2	2	$2 \times 2 \times 9 = 36$
10	10	0	0	$0 \times 0 \times 10 = 0$

Table 1.16: (continued)

Side of Square	Box Height	Box Width	Box Length	Volume
----------------	------------	-----------	------------	--------

We stop at a square of 10 inches because at this point we have cut out all of the cardboard and we can't make a box any more. From the table we see that we can make the biggest box if we cut out squares with a side length of three inches. This gives us a volume of 588 in^3 .

Answer The box of greatest volume is made if we cut out squares with a side length of three inches.

Step 4:

Check

We see that 588 in^3 is the largest volume appearing in the table. We picked integer values for the sides of the squares that we are cut out. Is it possible to get a larger value for the volume if we pick non-integer values? Since we get the largest volume for the side length equal to three inches, let's make another table with values close to three inches that is split into smaller increments:

Table 1.17:

Side of Square	Box Height	Box Width	Box Length	Volume
2.5	2.5	15	15	$15 \times 15 \times 2.5 = 562.5$
2.6	2.6	14.8	14.8	$14.8 \times 14.8 \times 2.6 = 569.5$
2.7	2.7	14.6	14.6	$14.6 \times 14.6 \times 2.7 = 575.5$
2.8	2.8	14.4	14.4	$14.4 \times 14.4 \times 2.8 = 580.6$
2.9	2.9	14.2	14.2	$14.2 \times 14.2 \times 2.9 = 584.8$
3	3	14	14	$14 \times 14 \times 3 = 588$
3.1	3.1	13.8	13.8	$13.8 \times 13.8 \times 3.1 = 590.4$
3.2	3.2	13.6	13.6	$13.6 \times 13.6 \times 3.2 = 591.9$
3.3	3.3	13.4	13.4	$13.4 \times 13.4 \times 3.3 = 592.5$
3.4	3.4	13.2	13.2	$13.2 \times 13.2 \times 3.4 = 592.4$
3.5	3.5	13	13	$13 \times 13 \times 3.5 = 591.5$

Notice that the largest volume is not when the side of the square is three inches, but rather when the side of the square is 3.3 inches.

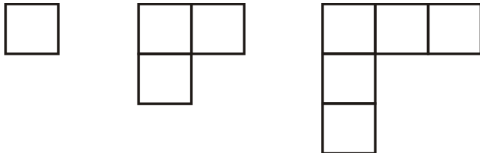
Our original answer was not incorrect, but it was not as accurate as it could be. We can get an even more accurate answer if we take even smaller increments of the side length of the square. To do that, we would choose smaller measurements that are in the neighborhood of 3.3 inches.

Meanwhile, our first answer checks out if we want it rounded to zero decimal places, but **a more accurate**

answer is 3.3 inches.

Review Questions

1. Go back and find the solution to the problem in Example 1.
2. Britt has \$2.25 in nickels and dimes. If she has 40 coins in total, how many of each coin does she have?
3. Jeremy divides a 160-square-foot garden into plots that are either 10 or 12 square feet each. If there are 14 plots in all, how many plots are there of each size?
4. A pattern of squares is put together as shown. How many squares are in the 12th diagram?



5. In Harrisville, local housing laws specify how many people can live in a house or apartment: the maximum number of people allowed is twice the number of bedrooms, plus one. If Jan, Pat, and their four children want to rent a house, how many bedrooms must it have?
6. A restaurant hosts children's birthday parties for a cost of \$120 for the first six children (including the birthday child) and \$30 for each additional child. If Jaden's parents have a budget of \$200 to spend on his birthday party, how many guests can Jaden invite?
7. A movie theater with 200 seats charges \$8 general admission and \$5 for students. If the 5:00 showing is sold out and the theater took in \$1468 for that showing, how many of the seats are occupied by students?
8. Oswald is trying to cut down on drinking coffee. His goal is to cut down to 6 cups per week. If he starts with 24 cups the first week, then cuts down to 21 cups the second week and 18 cups the third week, how many weeks will it take him to reach his goal?
9. Taylor checked out a book from the library and it is now 5 days late. The late fee is 10 cents per day. How much is the fine?
10. Mikhail is filling a sack with oranges.
 - (a) If each orange weighs 5 ounces and the sack will hold 2 pounds, how many oranges will the sack hold before it bursts?
 - (b) Mikhail plans to use these oranges to make breakfast smoothies. If each smoothie requires $\frac{3}{4}$ cup of orange juice, and each orange will yield half a cup, how many smoothies can he make?
11. Jessamyn takes out a \$150 loan from an agency that charges 12% of the original loan amount in interest each week. If she takes five weeks to pay off the loan, what is the total amount (loan plus interest) she will need to pay back?
12. How many hours will a car traveling at 75 miles per hour take to catch up to a car traveling at 55 miles per hour if the slower car starts two hours before the faster car?
13. Grace starts biking at 12 miles per hour. One hour later, Dan starts biking at 15 miles per hour, following the same route. How long will it take him to catch up with Grace?
14. A new theme park opens in Milford. On opening day, the park has 120 visitors; on each of the next three days, the park has 10 more visitors than the day before; and on each of the three days after that, the park has 20 more visitors than the day before.
 - (a) How many visitors does the park have on the seventh day?
 - (b) How many total visitors does the park have all week?
15. Lemuel wants to enclose a rectangular plot of land with a fence. He has 24 feet of fencing. What is the largest possible area that he could enclose with the fence?

16. Quizzes in Keiko's history class are worth 20 points each. Keiko scored 15 and 18 points on her last two quizzes. What score does she need on her third quiz to get an average score of 17 on all three?
17. Tickets to an event go on sale for \$20 six weeks before the event, and go up in price by \$5 each week. What is the price of tickets one week before the event?
18. Mark is three years older than Janet, and the sum of their ages is 15. How old are Mark and Janet?
19. In a one-on-one basketball game, Jane scored $1\frac{1}{2}$ times as many points as Russell. If the two of them together scored 10 points, how many points did Jane score?
20. Scientists are tracking two pods of whales during their migratory season. On the first day of June, one pod is 120 miles north of a certain group of islands, and every day thereafter it gets 15 miles closer to the islands. The second pod starts out 160 miles east of the islands on June 3, and heads toward the islands at a rate of 20 miles a day.
 - (a) Which pod will arrive at the islands first, and on what day?
 - (b) How long after that will it take the other pod to reach the islands?
 - (c) Suppose the pod that reaches the islands first immediately heads south from the islands at a rate of 15 miles a day, and the pod that gets there second also heads south from there at a rate of 25 miles a day. On what day will the second pod catch up with the first?
 - (d) How far will both pods be from the islands on that day?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9611>.

Chapter 2

Real Numbers

2.1 Integers and Rational Numbers

Learning Objectives

- Graph and compare integers.
- Classify and order rational numbers.
- Find opposites of numbers.
- Find absolute values.
- Compare fractions to determine which is bigger.

Introduction

One day, Jason leaves his house and starts walking to school. After three blocks, he stops to tie his shoe and leaves his lunch bag sitting on the curb. Two blocks farther on, he realizes his lunch is missing and goes back to get it. After picking up his lunch, he walks six more blocks to arrive at school. How far is the school from Jason's house? And how far did Jason actually walk to get there?

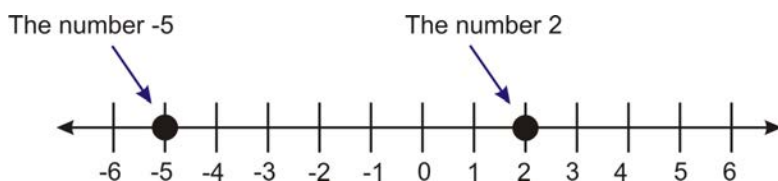
Graph and Compare Integers

Integers include the counting numbers (1, 2, 3, ...), the negative counting numbers (-1, -2, -3, ...), and zero. (You may have learned to think of integers as **whole numbers**, but technically the “whole numbers” don't include the negative integers.)

Example 1

Compare the numbers 2 and -5.

When we plot numbers on a number line, the **greatest** number is farthest to the right, and the **least** is farthest to the left.



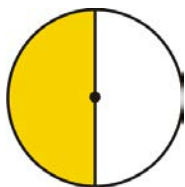
In the diagram above, we can see that 2 is farther to the right on the number line than -5, so we say that 2 is greater than -5. We use the symbol “>” to mean “greater than”, so we can write $2 > -5$.

Classifying Rational Numbers

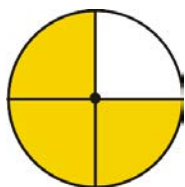
When we divide an integer a by another integer b (as long as b is not zero) we get a **rational number**. It's called this because it is the **ratio** of one number to another, and we can write it in fraction form as $\frac{a}{b}$. (You may recall that the top number in a fraction is called the **numerator** and the bottom number is called the **denominator**.)

You can think of a rational number as a fraction of a cake. If you cut the cake into b slices, your share is a of those slices.

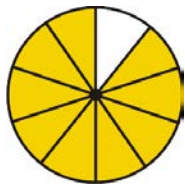
For example, when we see the rational number $\frac{1}{2}$, we can imagine cutting the cake into two parts. Our share is one of those parts. Visually, the rational number $\frac{1}{2}$ looks like this:



With the rational number $\frac{3}{4}$, we cut the cake into four parts and our share is three of those parts. Visually, the rational number $\frac{3}{4}$ looks like this:



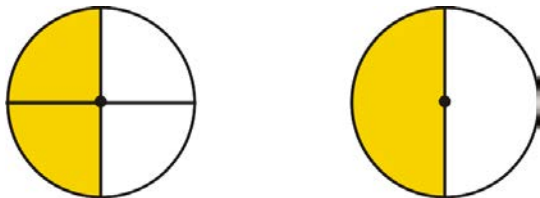
The rational number $\frac{9}{10}$ represents nine slices of a cake that has been cut into ten pieces. Visually, the rational number $\frac{9}{10}$ looks like this:



Proper fractions are rational numbers where the numerator is less than the denominator. A proper fraction represents a number less than one.

Improper fractions are rational numbers where the numerator is greater than or equal to the denominator. An improper fraction can be rewritten as a mixed number – an integer plus a proper fraction. For example, $\frac{9}{4}$ can be written as $2\frac{1}{4}$. An improper fraction represents a number greater than or equal to one.

Equivalent fractions are two fractions that represent the same amount. For example, look at a visual representation of the rational number $\frac{2}{4}$, and one of the number $\frac{1}{2}$.



You can see that the shaded regions are the same size, so the two fractions are equivalent. We can convert one fraction into the other by **reducing** the fraction, or writing it in lowest terms. To do this, we write out the prime factors of both the numerator and the denominator and cancel matching factors that appear in both the numerator **and** denominator.

$$\frac{2}{4} = \frac{2 \cdot 1}{2 \cdot 2 \cdot 1} = \frac{1}{2 \cdot 1} = \frac{1}{2}$$

Reducing a fraction doesn't change the value of the fraction—it just simplifies the way we write it. Once we've canceled all common factors, the fraction is in its **simplest form**.

Example 2

Classify and simplify the following rational numbers

- a) $\frac{3}{7}$
- b) $\frac{9}{3}$
- c) $\frac{50}{60}$

Solution

- a) 3 and 7 are both prime, so we can't factor them. That means $\frac{3}{7}$ is already in its simplest form. It is also a proper fraction.
- b) $\frac{9}{3}$ is an improper fraction because $9 > 3$. To simplify it, we factor the numerator and denominator and cancel: $\frac{3 \cdot 3}{3 \cdot 1} = \frac{3}{1} = 3$.
- c) $\frac{50}{60}$ is a proper fraction, and we can simplify it as follows: $\frac{50}{60} = \frac{5 \cdot 5 \cdot 2}{5 \cdot 3 \cdot 2 \cdot 2} = \frac{5}{3 \cdot 2} = \frac{5}{6}$.

Order Rational Numbers

Ordering rational numbers is simply a matter of arranging them by increasing value—least first and greatest last.

Example 3

Put the following fractions in order from least to greatest: $\frac{1}{2}$, $\frac{3}{4}$, $\frac{2}{3}$

Solution

$$\frac{1}{2} < \frac{2}{3} < \frac{3}{4}$$

Simple fractions are easy to order—we just know, for example, that one-half is greater than one quarter, and that two thirds is bigger than one-half. But how do we compare more complex fractions?

Example 4

Which is greater, $\frac{3}{7}$ or $\frac{4}{9}$?

In order to determine this, we need to rewrite the fractions so we can compare them more easily. If we rewrite them as equivalent fractions that have the same denominators, then we can compare them directly. To do this, we need to find the **lowest common denominator** (LCD), or the least common multiple of the two denominators.

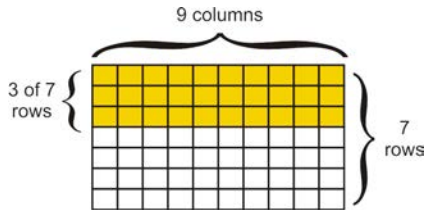
The lowest common multiple of 7 and 9 is 63. Our fraction will be represented by a shape divided into 63 sections. This time we will use a rectangle cut into 9 by 7 = 63 pieces.

7 divides into 63 nine times, so $\frac{3}{7} = \frac{9 \cdot 3}{9 \cdot 7} = \frac{27}{63}$.

We can multiply the numerator and the denominator both by 9 because that's really just the opposite of reducing the fraction—to get back from $\frac{27}{63}$ to $\frac{3}{7}$, we'd just cancel out the 9's. Or, to put that in more formal terms:

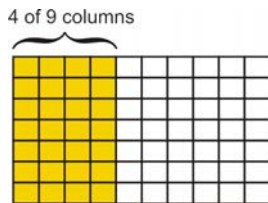
The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.

Therefore, $\frac{27}{63}$ is an equivalent fraction to $\frac{3}{7}$. Here it is shown visually:

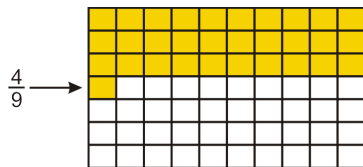


9 divides into 63 seven times, so $\frac{4}{9} = \frac{7 \cdot 4}{7 \cdot 9} = \frac{28}{63}$.

$\frac{28}{63}$ is an equivalent fraction to $\frac{4}{9}$. Here it is shown visually:



By writing the fractions with a **common denominator** of 63, we can easily compare them. If we take the 28 shaded boxes out of 63 (from our image of $\frac{4}{9}$ above) and arrange them in rows instead of columns, we can see that they take up more space than the 27 boxes from our image of $\frac{3}{7}$:



Solution

Since $\frac{28}{63}$ is greater than $\frac{27}{63}$, $\frac{4}{9}$ is greater than $\frac{3}{7}$.

Graph and Order Rational Numbers

To plot non-integer rational numbers (fractions) on the number line, we can convert them to mixed numbers (graphing is one of the few occasions in algebra when it's better to use mixed numbers than improper fractions), or we can convert them to decimal form.

Example 5

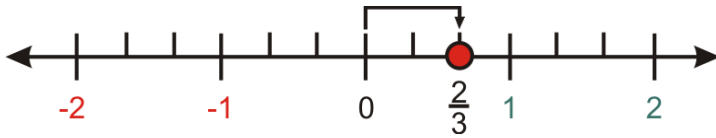
Plot the following rational numbers on the number line.

- $\frac{2}{3}$
- $-\frac{3}{7}$

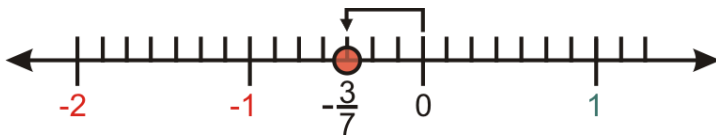
c) $\frac{17}{5}$

If we divide up the number line into sub-intervals based on the denominator of the fraction, we can look at the fraction's numerator to determine how many of these sub-intervals we need to include.

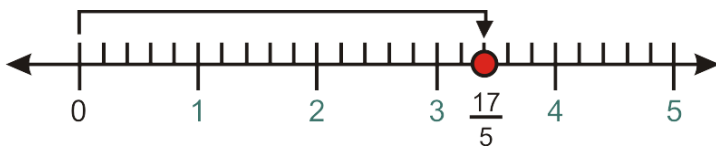
a) $\frac{2}{3}$ falls between 0 and 1. Because the denominator is 3, we divide the interval between 0 and 1 into three smaller units. Because the numerator is 2, we count two units over from 0.



b) $-\frac{3}{7}$ falls between 0 and -1. We divide the interval into seven units, and move left from zero by three of those units.



c) $\frac{17}{5}$ as a mixed number is $3\frac{2}{5}$ and falls between 3 and 4. We divide the interval into five units, and move over two units.

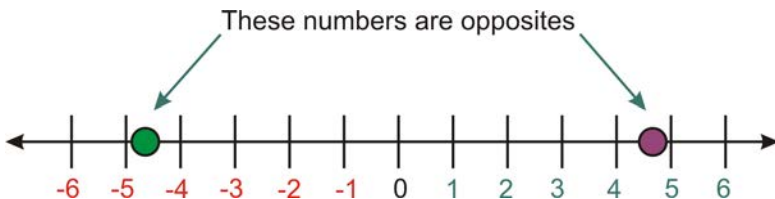


Another way to graph this fraction would be as a decimal. $3\frac{2}{5}$ is equal to 3.4, so instead of dividing the interval between 3 and 4 into 5 units, we could divide it into 10 units (each representing a distance of 0.1) and then count over 4 units. We would end up at the same place on the number line either way.

To make graphing rational numbers easier, try using the number line generator at <http://theworksheetgenerator.com/numline.html>. You can use it to create a number line divided into whatever units you want, as long as you express the units in decimal form.

Find the Opposites of Numbers

Every number has an opposite. On the number line, a number and its opposite are, predictably, *opposite* each other. In other words, they are the same distance from zero, but on opposite sides of the number line.



The opposite of zero is defined to be simply zero.

The sum of a number and its opposite is always zero—for example, $3 + -3 = 0$, $4.2 + -4.2 = 0$, and so on. This is because adding 3 and -3 is like moving 3 steps to the right along the number line, and then 3 steps back to the left. The number and its opposite cancel each other out, leaving zero.

Another way to think of the opposite of a number is that it is simply the original number multiplied by

-1. The opposite of 4 is 4×-1 or -4 , the opposite of -2.3 is -2.3×-1 or just 2.3 , and so on. Another term for the opposite of a number is the **additive inverse**.

Example 6

Find the opposite of each of the following:

d) 19.6

e) $-\frac{4}{9}$

f) x

g) xy^2

h) $(x - 3)$

Solution

Since we know that opposite numbers are on opposite sides of zero, we can simply multiply each expression by -1 . This changes the sign of the number to its opposite—if it's negative, it becomes positive, and vice versa.

a) The opposite of 19.6 is -19.6 .

b) The opposite of $-\frac{4}{9}$ is $\frac{4}{9}$.

c) The opposite of x is $-x$.

d) The opposite of xy^2 is $-xy^2$.

e) The opposite of $(x - 3)$ is $-(x - 3)$, or $(3 - x)$.

Note: With the last example you must multiply the **entire expression** by -1 . A common mistake in this example is to assume that the opposite of $(x - 3)$ is $(x + 3)$. Avoid this mistake!

Find Absolute Values

When we talk about absolute value, we are talking about distances on the number line. For example, the number 7 is 7 units away from zero—and so is the number -7 . The absolute value of a number is the distance it is from zero, so the absolute value of 7 and the absolute value of -7 are both 7.

We **write** the absolute value of -7 as $|-7|$. We **read** the expression $|x|$ as “the absolute value of x .”

- Treat absolute value expressions like parentheses. If there is an operation inside the absolute value symbols, evaluate that operation first.
- The absolute value of a number or an expression is **always** positive or zero. It cannot be negative. With absolute value, we are only interested in how far a number is from zero, and not in which direction.

Example 7

Evaluate the following absolute value expressions.

a) $|5 + 4|$

b) $3 - |4 - 9|$

c) $|-5 - 11|$

d) $-|7 - 22|$

(Remember to treat any expressions inside the absolute value sign as if they were inside parentheses, and evaluate them first.)

Solution

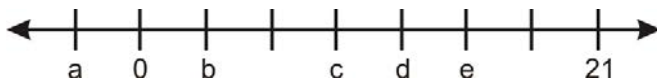
- a) $|5 + 4| = |9| = 9$
b) $3 - |4 - 9| = 3 - |-5| = 3 - 5 = -2$
c) $|-5 - 11| = |-16| = 16$
d) $-|7 - 22| = -|-15| = -(15) = -15$

Lesson Summary

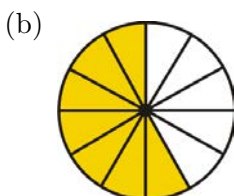
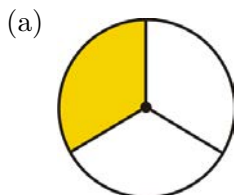
- **Integers** (or **whole numbers**) are the counting numbers (1, 2, 3, ...), the negative counting numbers (-1, -2, -3, ...), and zero.
- A **rational number** is the **ratio** of one integer to another, like $\frac{3}{5}$ or $\frac{a}{b}$. The top number is called the **numerator** and the bottom number (which can't be zero) is called the **denominator**.
- **Proper fractions** are rational numbers where the numerator is less than the denominator.
- **Improper fractions** are rational numbers where the numerator is greater than the denominator.
- **Equivalent fractions** are two fractions that equal the same numerical value. The fractions $\frac{a}{b}$ and $\frac{c \cdot a}{c \cdot b}$ are equivalent as long as $c \neq 0$.
- To **reduce** a fraction (write it in **simplest form**), write out all prime factors of the numerator and denominator, cancel common factors, then recombine.
- To compare two fractions it helps to write them with a **common denominator**.
- The **absolute value** of a number is the distance it is from zero on the number line. The absolute value of any expression will always be positive or zero.
- Two numbers are **opposites** if they are the same distance from zero on the number line and on opposite sides of zero. The opposite of an expression can be found by multiplying **the entire expression** by -1.

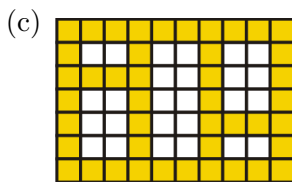
Review Questions

1. Solve the problem posed in the Introduction.
2. The tick-marks on the number line represent evenly spaced integers. Find the values of a, b, c, d and e .



3. Determine what fraction of the whole each shaded region represents.





4. Place the following sets of rational numbers in order, from least to greatest.

- (a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$
 (b) $\frac{1}{10}, \frac{1}{2}, \frac{2}{5}, \frac{1}{4}, \frac{7}{20}$
 (c) $\frac{39}{60}, \frac{49}{80}, \frac{59}{100}$
 (d) $\frac{7}{11}, \frac{8}{13}, \frac{12}{19}$
 (e) $\frac{9}{5}, \frac{22}{15}, \frac{4}{3}$

5. Find the simplest form of the following rational numbers.

- (a) $\frac{22}{44}$
 (b) $\frac{9}{27}$
 (c) $\frac{12}{18}$
 (d) $\frac{315}{420}$
 (e) $\frac{244}{168}$

6. Find the opposite of each of the following.

- (a) 1.001
 (b) $(5 - 11)$
 (c) $(x + y)$
 (d) $(x - y)$
 (e) $(x + y - 4)$
 (f) $(-x + 2y)$

7. Simplify the following absolute value expressions.

- (a) $11 - |-4|$
 (b) $|4 - 9| - |-5|$
 (c) $|-5 - 11|$
 (d) $7 - |22 - 15 - 19|$
 (e) $-|-7|$
 (f) $|-2 - 88| - |88 + 2|$

2.2 Adding and Subtracting Rational Numbers

Learning Objectives

- Add and subtract using a number line.
- Add and subtract rational numbers.
- Identify and apply properties of addition and subtraction.
- Solve real-world problems using addition and subtraction of fractions.
- Evaluate change using a variable expression.

Introduction

Ilana buys two identically sized cakes for a party. She cuts the chocolate cake into 24 pieces and the vanilla cake into 20 pieces, and lets the guests serve themselves. Martin takes three pieces of chocolate cake and

one of vanilla, and Sheena takes one piece of chocolate and two of vanilla. Which of them gets more cake?

Add and Subtract Using a Number Line

In Lesson 1, we learned how to represent numbers on a number line. To add numbers on a number line, we start at the position of the first number, and then move to the right by a number of units equal to the second number.

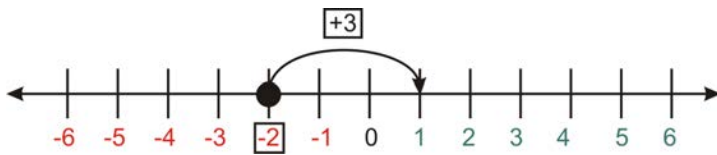
Example 1

Represent the sum $-2 + 3$ on a number line.

We start at the number -2 , and then move 3 units to the right. We thus end at $+1$.

Solution

$$-2 + 3 = 1$$



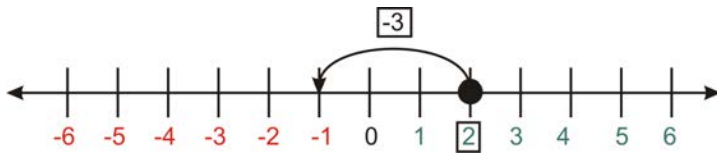
Example 2

Represent the sum $2 - 3$ on a number line.

Subtracting a number is basically just **adding a negative number**. Instead of moving to the right, we move to the left. Starting at the number 2, and then moving 3 to the left, means we end at -1 .

Solution

$$2 - 3 = -1$$



Adding and Subtracting Rational Numbers

When we add or subtract two fractions, the denominators must match before we can find the sum or difference. We have already seen how to find a common denominator for two rational numbers.

Example 3

Simplify $\frac{3}{5} + \frac{1}{6}$.

To combine these fractions, we need to rewrite them over a common denominator. We are looking for the **lowest common denominator** (LCD). We need to identify the **lowest common multiple** or **least common multiple** (LCM) of 5 and 6. That is the smallest number that both 5 and 6 divide into evenly (that is, without a remainder).

The lowest number that 5 and 6 both divide into evenly is 30. The LCM of 5 and 6 is 30, so the lowest common denominator for our fractions is also 30.

We need to rewrite our fractions as new **equivalent fractions** so that the denominator in each case is 30.

If you think back to our idea of a cake cut into a number of slices, $\frac{3}{5}$ means 3 slices of a cake that has been

cut into 5 pieces. You can see that if we cut the same cake into 30 pieces (6 times as many) we would need 6 times as many slices to make up an equivalent fraction of the cake—in other words, 18 slices instead of 3.



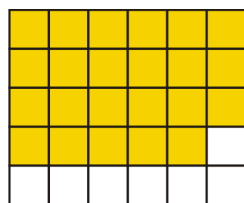
$\frac{3}{5}$ is equivalent to $\frac{18}{30}$.

By a similar argument, we can rewrite the fraction $\frac{1}{6}$ as a share of a cake that has been cut into 30 pieces. If we cut it into 5 times as many pieces, we need 5 times as many slices.



$\frac{1}{6}$ is equivalent to $\frac{5}{30}$.

Now that both fractions have the same denominator, we can add them. If we add 18 pieces of cake to 5 pieces, we get a total of 23 pieces. 23 pieces of a cake that has been cut into 30 pieces means that our answer is $\frac{23}{30}$.



$$\frac{3}{5} + \frac{1}{6} = \frac{18}{30} + \frac{5}{30} = \frac{23}{30}$$

Notice that when we have fractions with a common denominator, we **add the numerators** but we **leave the denominators alone**. Here is this information in algebraic terms.

When adding fractions: $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$

Example 4

Simplify $\frac{1}{3} - \frac{1}{9}$.

The lowest common multiple of 9 and 3 is 9, so 9 is our common denominator. That means we don't have to alter the second fraction at all.

3 divides into 9 three times, so $\frac{1}{3} = \frac{3 \cdot 1}{3 \cdot 3} = \frac{3}{9}$. Our sum becomes $\frac{3}{9} - \frac{1}{9}$. We can subtract fractions with a common denominator by subtracting their numerators, just like adding. In other words:

When subtracting fractions: $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$

Solution

$$\frac{1}{3} - \frac{1}{9} = \frac{2}{9}$$

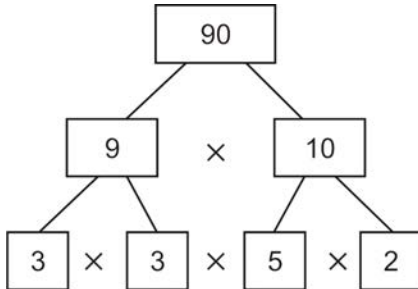
So far, we've only dealt with examples where it's easy to find the least common multiple of the denominators. With larger numbers, it isn't so easy to be sure that we have the LCD. We need a more systematic method. In the next example, we will use the method of **prime factors** to find the least common denominator.

Example 5

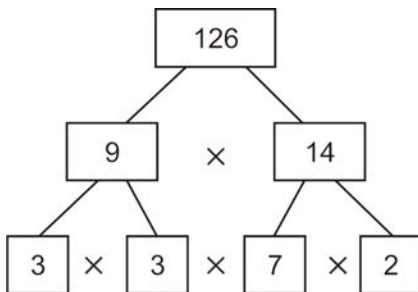
Simplify $\frac{29}{90} - \frac{13}{126}$.

To find the lowest common multiple of 90 and 126, we first find the prime factors of 90 and 126. We do this by continually dividing the number by factors until we can't divide any further. You may have seen a factor tree before. (For practice creating factor trees, try the Factor Tree game at http://www.mathgoodies.com/factors/factor_tree.asp.)

The factor tree for 90 looks like this:



The factor tree for 126 looks like this:



The LCM for 90 and 126 is made from the **smallest possible collection of primes** that enables us to construct either of the two numbers. We take only enough instances of each prime to make the number with the greater number of instances of that prime in its factor tree.

Table 2.1:

Prime	Factors in 90	Factors in 126	We Need
2	1	1	1
3	2	2	2
5	1	0	1
7	0	1	1

So we need one 2, two 3's, one 5 and one 7. That gives us $2 \cdot 3 \cdot 3 \cdot 5 \cdot 7 = 630$ as the lowest common multiple of 90 and 126. So 630 is the LCD for our calculation.

90 divides into 630 seven times (notice that 7 is the only factor in 630 that is missing from 90), so $\frac{29}{90} = \frac{7 \cdot 29}{7 \cdot 90} = \frac{203}{630}$.

126 divides into 630 five times (notice that 5 is the only factor in 630 that is missing from 126), so $\frac{13}{126} = \frac{5 \cdot 13}{5 \cdot 126} = \frac{65}{630}$.

Now we complete the problem: $\frac{29}{90} - \frac{13}{126} = \frac{203}{630} - \frac{65}{630} = \frac{138}{630}$.

This fraction **simplifies**. To be sure of finding the **simplest form** for $\frac{138}{630}$, we write out the prime factors of the numerator and denominator. We already know the prime factors of 630. The prime factors of 138

are 2, 3 and 23.

$\frac{138}{630} = \frac{2 \cdot 3 \cdot 23}{2 \cdot 3 \cdot 3 \cdot 5 \cdot 7}$; one factor of 2 and one factor of 3 cancels out, leaving $\frac{23}{3 \cdot 5 \cdot 7}$ or $\frac{23}{105}$ as our answer.

Identify and Apply Properties of Addition

Three mathematical properties which involve addition are the **commutative**, **associative**, and the **additive identity properties**.

Commutative property: When two numbers are added, the sum is the same even if the order of the items being added changes.

Example: $3 + 2 = 2 + 3$

Associative Property: When three or more numbers are added, the sum is the same regardless of how they are grouped.

Example: $(2 + 3) + 4 = 2 + (3 + 4)$

Additive Identity Property: The sum of any number and zero is the original number.

Example: $5 + 0 = 5$

Solve Real-World Problems Using Addition and Subtraction

Example 6

Peter is hoping to travel on a school trip to Europe. The ticket costs \$2400. Peter has several relatives who have pledged to help him with the ticket cost. His parents have told him that they will cover half the cost. His grandma Zenoviea will pay one sixth, and his grandparents in Florida will send him one fourth of the cost. What fraction of the cost can Peter count on his relatives to provide?

The first thing we need to do is extract the relevant information. Peter's parents will provide $\frac{1}{2}$ the cost; his grandma Zenoviea will provide $\frac{1}{6}$; and his grandparents in Florida $\frac{1}{4}$. We need to find the sum of those numbers, or $\frac{1}{2} + \frac{1}{6} + \frac{1}{4}$.

To determine the sum, we first need to find the LCD. The LCM of 2, 6 and 4 is 12, so that's our LCD. Now we can find equivalent fractions:

$$\begin{aligned}\frac{1}{2} &= \frac{6 \cdot 1}{6 \cdot 2} = \frac{6}{12} \\ \frac{1}{6} &= \frac{2 \cdot 1}{2 \cdot 6} = \frac{2}{12} \\ \frac{1}{4} &= \frac{3 \cdot 1}{3 \cdot 4} = \frac{3}{12}\end{aligned}$$

Putting them all together: $\frac{6}{12} + \frac{2}{12} + \frac{3}{12} = \frac{11}{12}$.

Peter will get $\frac{11}{12}$ the cost of the trip, or \$2200 out of \$2400, from his family.

Example 7

A property management firm is buying parcels of land in order to build a small community of condominiums. It has just bought three adjacent plots of land. The first is four-fifths of an acre, the second is five-twelfths of an acre, and the third is nineteen-twentieths of an acre. The firm knows that it must allow one-sixth of an acre for utilities and a small access road. How much of the remaining land is available for development?

The first thing we need to do is extract the relevant information. The plots of land measure $\frac{4}{5}$, $\frac{5}{12}$, and $\frac{19}{20}$ acres, and the firm can use all of that land except for $\frac{1}{6}$ of an acre. The total amount of land the firm can

use is therefore $\frac{4}{5} + \frac{5}{12} + \frac{19}{20} - \frac{1}{6}$ acres.

We can add and subtract multiple fractions at once just by finding a common denominator for all of them. The factors of 5, 9, 20, and 6 are as follows:

$$\begin{array}{rcl} 5 & & 5 \\ 12 & & 2 \cdot 2 \cdot 3 \\ 20 & & 2 \cdot 2 \cdot 5 \\ 6 & & 2 \cdot 3 \end{array}$$

We need a 5, two 2's, and a 3 in our LCD. $2 \cdot 2 \cdot 3 \cdot 5 = 60$, so that's our common denominator. Now to convert the fractions:

$$\begin{array}{rcl} \frac{4}{5} & = & \frac{12 \cdot 4}{12 \cdot 5} = \frac{48}{60} \\ \frac{5}{12} & = & \frac{5 \cdot 5}{5 \cdot 12} = \frac{25}{60} \\ \frac{19}{20} & = & \frac{3 \cdot 19}{3 \cdot 20} = \frac{57}{60} \\ \frac{1}{6} & = & \frac{10 \cdot 1}{10 \cdot 6} = \frac{10}{60} \end{array}$$

We can rewrite our sum as $\frac{48}{60} + \frac{25}{60} + \frac{57}{60} - \frac{10}{60} = \frac{48+25+57-10}{60} = \frac{120}{60}$.

Next, we need to reduce this fraction. We can see immediately that the numerator is twice the denominator, so this fraction reduces to $\frac{2}{1}$ or simply 2. One is sometimes called the **invisible denominator**, because every whole number can be thought of as a rational number whose denominator is one.

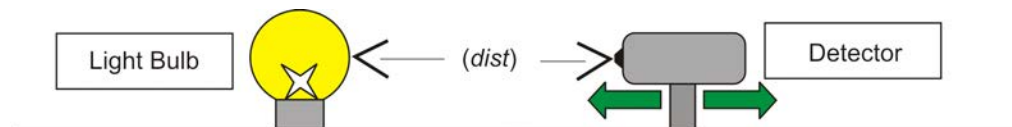
Solution

The property firm has two acres available for development.

Evaluate Change Using a Variable Expression

When we write algebraic expressions to represent a real quantity, the difference between two values is the **change** in that quantity.

Example 8



The intensity of light hitting a detector when it is held a certain distance from a bulb is given by this equation:

$$\text{Intensity} = \frac{3}{d^2}$$

where d is the distance measured in **meters**, and intensity is measured in **lumens**. Calculate the change in intensity when the detector is moved from two meters to three meters away.

We first find the values of the intensity at distances of two and three meters.

$$\text{Intensity (2)} = \frac{3}{(2)^2} = \frac{3}{4}$$

$$\text{Intensity (3)} = \frac{3}{(3)^2} = \frac{3}{9} = \frac{1}{3}$$

The **difference** in the two values will give the **change** in the intensity. We move **from** two meters **to** three meters away.

$$\text{Change} = \text{Intensity (3)} - \text{Intensity (2)} = \frac{1}{3} - \frac{3}{4}$$

To find the answer, we will need to write these fractions over a common denominator.

The LCM of 3 and 4 is 12, so we need to rewrite each fraction with a denominator of 12:

$$\frac{1}{3} = \frac{4 \cdot 1}{4 \cdot 3} = \frac{4}{12}$$
$$\frac{3}{4} = \frac{3 \cdot 3}{3 \cdot 4} = \frac{9}{12}$$

So we can rewrite our equation as $\frac{4}{12} - \frac{9}{12} = -\frac{5}{12}$. The negative value means that the intensity decreases as we move from 2 to 3 meters away.

Solution

When moving the detector from two meters to three meters, the intensity falls by $\frac{5}{12}$ lumens.

Lesson Summary

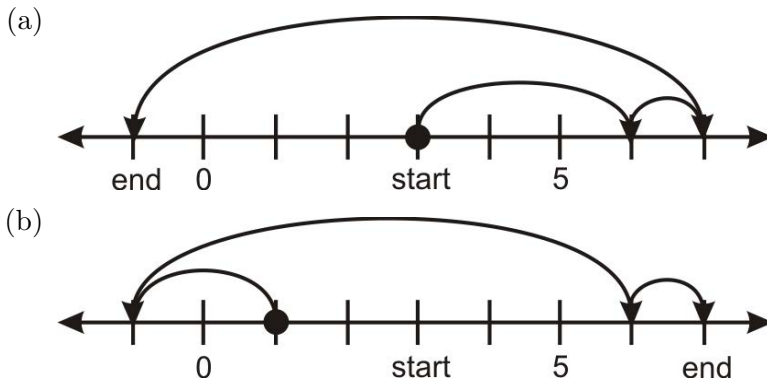
- **Subtracting** a number is the same as adding the **opposite** (or **additive inverse**) of the number.
- To add fractions, rewrite them over the **lowest common denominator (LCD)**. The lowest common denominator is the **lowest** (or **least**) **common multiple (LCM)** of the two denominators.
- When **adding fractions**: $\frac{a}{c} + \frac{b}{c} = \frac{a+b}{c}$
- When **subtracting fractions**: $\frac{a}{c} - \frac{b}{c} = \frac{a-b}{c}$
- **Commutative property**: the sum of two numbers is the same even if the order of the items to be added changes.
- **Associative Property**: When three or more numbers are added, the sum is the same regardless of how they are grouped.
- **Additive Identity Property**: The sum of any number and zero is the original number.
- The number one is sometimes called the **invisible denominator**, as every whole number can be thought of as a rational number whose denominator is one.
- The **difference** between two values is the **change** in that quantity.

Further Practice

For more practice adding and subtracting fractions, try playing the math games at http://www.mathplayground.com/fractions_add.html and http://www.mathplayground.com/fractions_sub.html, or the one at <http://www.aaamath.com/fra66kx2.htm>.

Review Questions

1. Write the sum that the following moves on a number line represent.



2. Add the following rational numbers. Write each answer in its **simplest form**.

- (a) $\frac{3}{7} + \frac{2}{7}$
 (b) $\frac{3}{10} + \frac{1}{5}$
 (c) $\frac{5}{16} + \frac{5}{12}$
 (d) $\frac{3}{8} + \frac{9}{16}$
 (e) $\frac{8}{25} + \frac{7}{10}$
 (f) $\frac{1}{6} + \frac{1}{4}$
 (g) $\frac{7}{15} + \frac{2}{9}$
 (h) $\frac{5}{19} + \frac{2}{27}$

3. Which property of addition does each situation involve?

- (a) Whichever order your groceries are scanned at the store, the total will be the same.
 (b) However many shovel-loads it takes to move 1 ton of gravel, the number of rocks moved is the same.
 (c) If Julia has no money, then Mark and Julia together have just as much money as Mark by himself has.

4. Solve the problem posed in the Introduction to this lesson.

5. Nadia, Peter and Ian are pooling their money to buy a gallon of ice cream. Nadia is the oldest and gets the greatest allowance. She contributes half of the cost. Ian is next oldest and contributes one third of the cost. Peter, the youngest, gets the smallest allowance and contributes one fourth of the cost. They figure that this will be enough money. When they get to the check-out, they realize that they forgot about sales tax and worry there will not be enough money. Amazingly, they have exactly the right amount of money. What fraction of the cost of the ice cream was added as tax?

6. Subtract the following rational numbers. Be sure that your answer is in the **simplest form**.

- (a) $\frac{5}{12} - \frac{9}{18}$
 (b) $\frac{3}{5} - \frac{1}{4}$
 (c) $\frac{3}{4} - \frac{1}{3}$
 (d) $\frac{1}{11} - \frac{9}{7}$
 (e) $\frac{2}{13} - \frac{1}{11}$
 (f) $\frac{7}{27} - \frac{9}{39}$
 (g) $\frac{6}{11} - \frac{3}{22}$
 (h) $\frac{13}{64} - \frac{7}{40}$
 (i) $\frac{11}{70} - \frac{11}{30}$

7. Consider the equation $y = 3x + 2$. Determine the change in y between $x = 3$ and $x = 7$.

8. Consider the equation $y = \frac{2}{3}x + \frac{1}{2}$. Determine the change in y between $x = 1$ and $x = 2$.

9. The time taken to commute from San Diego to Los Angeles is given by the equation $time = \frac{120}{speed}$ where $time$ is measured in **hours** and $speed$ is measured in **miles per hour** (mph). Calculate the change

in time that a rush hour commuter would see when switching from traveling by bus to traveling by train, if the bus averages 40 mph and the train averages 90 mph.

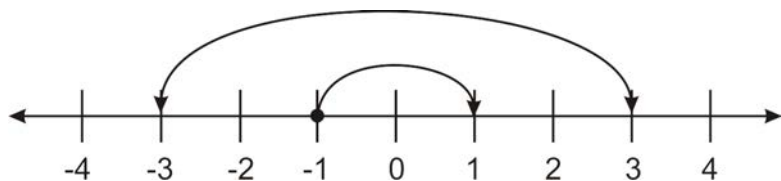
2.3 Multiplying and Dividing Rational Numbers

Learning Objectives

- Multiply by negative one.
- Multiply rational numbers.
- Identify and apply properties of multiplication.
- Solve real-world problems using multiplication.
- Find multiplicative inverses.
- Divide rational numbers.
- Solve real-world problems using division.

Multiplying Numbers by Negative One

Whenever we multiply a number by negative one, the sign of the number changes. In more mathematical terms, multiplying by negative one maps a number onto its opposite. The number line below shows two examples: $3 \cdot -1 = -3$ and $-1 \cdot -1 = 1$.



When we multiply a number by negative one, the absolute value of the new number is the same as the absolute value of the old number, since both numbers are the same distance from zero.

The product of a number “ x ” and negative one is $-x$. This does not mean that $-x$ is necessarily less than zero! If x itself is negative, then $-x$ will be positive because a negative times a negative (negative one) is a positive.

When you multiply an expression by negative one, remember to multiply the **entire expression** by negative one.

Example 1

Multiply the following by negative one.

- 79.5
- π
- $(x + 1)$
- $|x|$

Solution

- 79.5
- $-\pi$
- $-(x + 1)$ or $-x - 1$
- $-|x|$

Note that in the last case the negative sign **outside** the absolute value symbol applies **after** the absolute value. Multiplying the **argument** of an absolute value equation (the term inside the absolute value symbol) does not change the absolute value. $|x|$ is always positive. $| -x|$ is always positive. $-|x|$ is always negative.

Whenever you are working with expressions, you can check your answers by substituting in numbers for the variables. For example, you could check part *d* of Example 1 by letting $x = -3$. Then you'd see that $| -3| \neq -|3|$, because $| -3| = 3$ and $-|3| = -3$.

Careful, though—plugging in numbers can tell you if your answer is wrong, but it won't always tell you for sure if your answer is right!

Multiply Rational Numbers

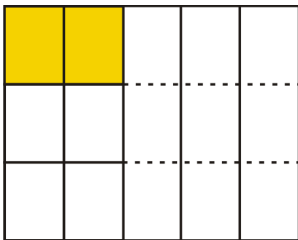
Example 2

Simplify $\frac{1}{3} \cdot \frac{2}{5}$.

One way to solve this is to think of money. For example, we know that *one third of sixty dollars* is written as $\frac{1}{3} \cdot \$60$. We can read the above problem as *one-third of two-fifths*. Here is a visual picture of the fractions *one-third* and *two-fifths*.



If we divide our rectangle into thirds one way and fifths the other way, here's what we get:



Here is the intersection of the two shaded regions. The whole has been divided into five pieces width-wise and three pieces height-wise. We get two pieces out of a total of fifteen pieces.

Solution

$$\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$$

Notice that $1 \cdot 2 = 2$ and $3 \cdot 5 = 15$. This turns out to be true in general: when you multiply rational numbers, the numerators multiply together and the denominators multiply together. Or, to put it more formally:

When multiplying fractions: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

This rule doesn't just hold for the product of two fractions, but for any number of fractions.

Example 4

Multiply the following rational numbers:

a) $\frac{2}{5} \cdot \frac{5}{9}$

b) $\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5}$

c) $\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5}$

Solution

a) With this problem, we can cancel the fives: $\frac{2}{5} \cdot \frac{5}{9} = \frac{2\cancel{5}}{\cancel{5}9} = \frac{2}{9}$.

b) With this problem, we multiply **all the numerators** and **all the denominators**:

$$\frac{1}{3} \cdot \frac{2}{7} \cdot \frac{2}{5} = \frac{1 \cdot 2 \cdot 2}{3 \cdot 7 \cdot 5} = \frac{4}{105}$$

c) With this problem, we multiply all the numerators and all the denominators, and then we can cancel most of them. The 2's, 3's, and 4's all cancel out, leaving $\frac{1}{5}$.

With multiplication of fractions, we can simplify before or after we multiply. The next example uses factors to help simplify before we multiply.

Example 5

Evaluate and simplify $\frac{12}{25} \cdot \frac{35}{42}$.

Solution

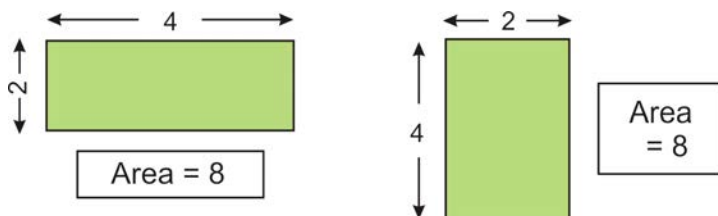
We can see that 12 and 42 are both multiples of six, 25 and 35 are both multiples of five, and 35 and 42 are both multiples of 7. That means we can write the whole product as $\frac{6\cancel{2}}{\cancel{5}5} \cdot \frac{\cancel{5}7}{\cancel{6}7} = \frac{6\cancel{2}\cancel{5}7}{\cancel{5}\cancel{5}\cancel{6}7}$. Then we can cancel out the 5, the 6, and the 7, leaving $\frac{2}{5}$.

Identify and Apply Properties of Multiplication

The four mathematical properties which involve multiplication are the **Commutative**, **Associative**, **Multiplicative Identity** and **Distributive Properties**.

Commutative property: When two numbers are multiplied together, the product is the same regardless of the order in which they are written.

Example: $4 \cdot 2 = 2 \cdot 4$



We can see a geometrical interpretation of **The Commutative Property of Multiplication** to the right. The Area of the shape (length \times width) is the same no matter which way we draw it.

Associative Property: When three or more numbers are multiplied, the product is the same regardless of their grouping.

Example: $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$

Multiplicative Identity Property: The product of one and any number is that number.

Example: $5 \cdot 1 = 5$

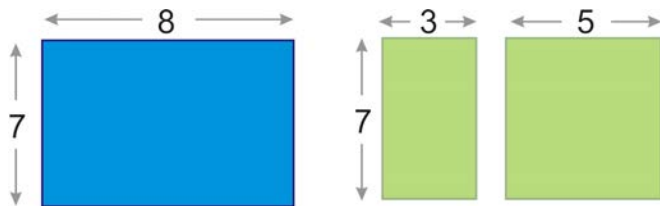
Distributive property: The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.

Example: $4(6 + 3) = 4 \cdot 6 + 4 \cdot 3$

Example 6

A gardener is planting vegetables for the coming growing season. He wishes to plant potatoes and has a

choice of a single 8×7 meter plot, or two smaller plots of 3×7 and 5×7 meters. Which option gives him the largest area for his potatoes?



Solution

In the first option, the gardener has a total area of (8×7) or 56 square meters.

In the second option, the gardener has (3×7) or 21 square meters, plus (5×7) or 35 square meters. $21 + 35 = 56$, so the area is the same as in the first option.

Solve Real-World Problems Using Multiplication

Example 7

In the chemistry lab there is a bottle with two liters of a 15% solution of hydrogen peroxide (H_2O_2). John removes one-fifth of what is in the bottle, and puts it in a beaker. He measures the amount of H_2O_2 and adds twice that amount of water to the beaker. Calculate the following measurements.

- The amount of H_2O_2 left in the bottle.
- The amount of diluted H_2O_2 in the beaker.
- The concentration of the H_2O_2 in the beaker.

Solution

a) To determine the amount of H_2O_2 left in the bottle, we first determine the amount that was removed. That amount was $\frac{1}{5}$ of the amount in the bottle (2 liters). $\frac{1}{5}$ of 2 is $\frac{2}{5}$.

The amount remaining is $2 - \frac{2}{5}$, or $\frac{10}{5} - \frac{2}{5} = \frac{8}{5}$ liter (or 1.6 liters).

There are 1.6 liters left in the bottle.

b) We determined that the amount of the 15% H_2O_2 solution removed was $\frac{2}{5}$ liter. The amount of water added was twice this amount, or $\frac{4}{5}$ liter. So the total amount of solution in the beaker is now $\frac{2}{5} + \frac{4}{5} = \frac{6}{5}$ liter, or 1.2 liters.

There are 1.2 liters of diluted H_2O_2 in the beaker.

c) The new concentration of H_2O_2 can be calculated.

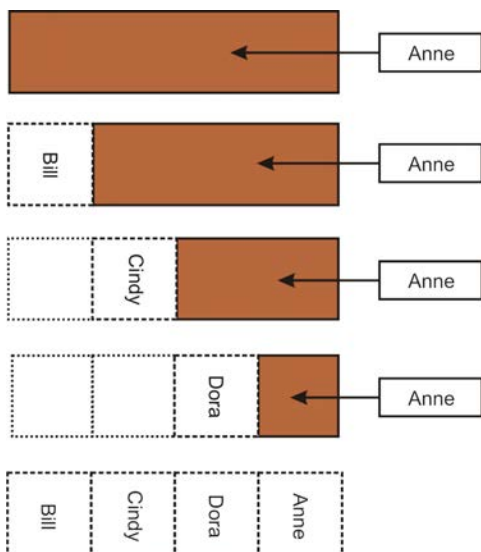
John started with $\frac{2}{5}$ liter of 15% H_2O_2 solution, so the amount of **pure** H_2O_2 is 15% of $\frac{2}{5}$ liters, or $0.15 \times 0.40 = 0.06$ liters.

After he adds the water, there is 1.2 liters of solution in the beaker, so the concentration of H_2O_2 is $\frac{0.06}{1.2} = \frac{1}{20}$ or 0.05. To convert to a percent we multiply this number by 100, so the beaker's contents are 5% H_2O_2 .

Example 8

Anne has a bar of chocolate and she offers Bill a piece. Bill quickly breaks off $\frac{1}{4}$ of the bar and eats it. Another friend, Cindy, takes $\frac{1}{3}$ of what was left. Anne splits the remaining candy bar into two equal pieces which she shares with a third friend, Dora. How much of the candy bar does each person get?

First, let's look at this problem visually.



Anne starts with a full candy bar.

Bill breaks off $\frac{1}{4}$ of the bar.

Cindy takes $\frac{1}{3}$ of what was left.

Dora gets half of the remaining candy bar.

We can see that the candy bar ends up being split four ways, with each person getting an equal amount.

Solution

Each person gets exactly $\frac{1}{4}$ of the candy bar.

We can also examine this problem using rational numbers. We keep a running total of what fraction of the bar remains. Remember, when we read a fraction followed by *of* in the problem, it means we multiply by that fraction.

We start with 1 bar. Then Bill takes $\frac{1}{4}$ of it, so there is $1 - \frac{1}{4} = \frac{3}{4}$ of a bar left.

Cindy takes $\frac{1}{3}$ of what's left, or $\frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ of a whole bar. That leaves $\frac{3}{4} - \frac{1}{4} = \frac{2}{4}$, or $\frac{1}{2}$ of a bar.

That half bar gets split between Anne and Dora, so they each get half of a half bar: $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

So each person gets exactly $\frac{1}{4}$ of the candy bar.

Extension: If each person's share is 3 oz, how much did the original candy bar weigh?

Identity Elements

An **identity element** is a number which, when combined with a mathematical operation on a number, leaves that number unchanged. For example, the **identity element** for addition and subtraction is **zero**, because adding or subtracting zero to a number doesn't change the number. And zero is also what you get when you add together a number and its opposite, like 3 and -3.

The **inverse operation** of addition is subtraction—when you add a number and then subtract that same number, you end up back where you started. Also, adding a number's opposite is the same as subtracting it—for example, $4 + (-3)$ is the same as $4 - 3$.

Multiplication and division are also inverse operations to each other—when you multiply by a number and then divide by the same number, you end up back where you started. Multiplication and division also have an identity element: when you multiply or divide a number by **one**, the number doesn't change.

Just as the **opposite** of a number is the number you can add to it to get zero, the **reciprocal** of a number is the number you can multiply it by to get one. And finally, just as adding a number's opposite is the same as subtracting the number, multiplying by a number's reciprocal is the same as dividing by the number.

Find Multiplicative Inverses

The reciprocal of a number x is also called the **multiplicative inverse**. Any number times its own multiplicative inverse equals one, and the multiplicative inverse of x is written as $\frac{1}{x}$.

To find the multiplicative inverse of a rational number, we simply *invert the fraction*—that is, flip it over. In other words:

The multiplicative inverse of $\frac{a}{b}$ is $\frac{b}{a}$, as long as $a \neq 0$.

You'll see why in the following exercise.

Example 9

Find the multiplicative inverse of each of the following.

a) $\frac{3}{7}$

b) $\frac{4}{9}$

c) $3\frac{1}{2}$

d) $-\frac{x}{y}$

e) $\frac{1}{11}$

Solution

a) When we invert the fraction $\frac{3}{7}$, we get $\frac{7}{3}$. Notice that if we multiply $\frac{3}{7} \cdot \frac{7}{3}$, the 3's and the 7's both cancel out and we end up with $\frac{1}{1}$, or just 1.

b) Similarly, the inverse of $\frac{4}{9}$ is $\frac{9}{4}$; if we multiply those two fractions together, the 4's and the 9's cancel out and we're left with 1. That's why the rule "invert the fraction to find the multiplicative inverse" works: the numerator and the denominator always end up canceling out, leaving 1.

c) To find the multiplicative inverse of $3\frac{1}{2}$ we first need to convert it to an improper fraction. Three wholes is six halves, so $3\frac{1}{2} = \frac{6}{2} + \frac{1}{2} = \frac{7}{2}$. That means the inverse is $\frac{2}{7}$.

d) Don't let the negative sign confuse you. The multiplicative inverse of a negative number is also negative! Just ignore the negative sign and flip the fraction as usual.

The multiplicative inverse of $-\frac{x}{y}$ is $-\frac{y}{x}$.

e) The multiplicative inverse of $\frac{1}{11}$ is $\frac{11}{1}$, or simply 11.

Look again at the last example. When we took the multiplicative inverse of $\frac{1}{11}$ we got a whole number, 11. That's because we can treat that whole number like a fraction with a denominator of 1. Any number, even a non-rational one, can be treated this way, so we can always find a number's multiplicative inverse using the same method.

Divide Rational Numbers

Earlier, we mentioned that multiplying by a number's reciprocal is the same as dividing by the number. That's how we can divide rational numbers; to divide by a rational number, just multiply by that number's reciprocal. In more formal terms:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}.$$

Example 10

Divide the following rational numbers, giving your answer in the **simplest form**.

a) $\frac{1}{2} \div \frac{1}{4}$

b) $\frac{7}{3} \div \frac{2}{3}$

c) $\frac{x}{2} \div \frac{1}{4y}$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right)$

Solution

a) Replace $\frac{1}{4}$ with $\frac{4}{1}$ and multiply: $\frac{1}{2} \times \frac{4}{1} = \frac{4}{2} = 2$.

b) Replace $\frac{2}{3}$ with $\frac{3}{2}$ and multiply: $\frac{7}{3} \times \frac{3}{2} = \frac{7 \cdot 3}{3 \cdot 2} = \frac{7}{2}$.

c) $\frac{x}{2} \div \frac{1}{4y} = \frac{x}{2} \times \frac{4y}{1} = \frac{4xy}{2} = \frac{2xy}{1} = 2xy$

d) $\frac{11}{2x} \div \left(-\frac{x}{y}\right) = \frac{11}{2x} \times \left(-\frac{y}{x}\right) = -\frac{11y}{2x^2}$

Solve Real-World Problems Using Division

Speed, Distance and Time

An object moving at a certain **speed** will cover a fixed **distance** in a set **time**. The quantities speed, distance and time are related through the equation $\text{Speed} = \frac{\text{Distance}}{\text{Time}}$.

Example 11

Andrew is driving down the freeway. He passes mile marker 27 at exactly mid-day. At 12:35 he passes mile marker 69. At what speed, in miles per hour, is Andrew traveling?

Solution

To find the speed, we need the distance traveled and the time taken. If we want our speed to come out in miles per hour, we'll need distance in **miles** and time in **hours**.

The distance is $69 - 27$ or 42 miles. The time is 35 minutes, or $\frac{35}{60}$ hours, which reduces to $\frac{7}{12}$. Now we can *plug in* the values for distance and time into our equation for speed.

$$\text{Speed} = \frac{42}{\frac{7}{12}} = 42 \div \frac{7}{12} = \frac{42}{1} \times \frac{12}{7} = \frac{6 \cdot 7 \cdot 12}{1 \cdot 7} = \frac{6 \cdot 12}{1} = 72$$

Andrew is driving at 72 miles per hour.

Example 12

Anne runs a mile and a half in a quarter hour. What is her speed in miles per hour?

Solution

We already have the distance and time in the correct units (miles and hours), so we just need to write them as fractions and plug them into the equation.

$$\text{Speed} = \frac{1\frac{1}{2}}{\frac{1}{4}} = \frac{3}{2} \div \frac{1}{4} = \frac{3}{2} \times \frac{4}{1} = \frac{3 \cdot 4}{2 \cdot 1} = \frac{12}{2} = 6$$

Anne runs at 6 miles per hour.

Example 13 – Newton’s Second Law

Newton’s second law ($F = ma$) relates the force applied to a body in Newtons (F), the mass of the body in kilograms (m) and the acceleration in meters per second squared (a). Calculate the resulting acceleration if a Force of $7\frac{1}{3}$ Newtons is applied to a mass of $\frac{1}{5}$ kg.

Solution

First, we rearrange our equation to isolate the acceleration, a . If $F = ma$, dividing both sides by m gives us $a = \frac{F}{m}$. Then we substitute in the known values for F and m :

$$a = \frac{7\frac{1}{3}}{\frac{1}{5}} = \frac{22}{3} \div \frac{1}{5} = \frac{22}{3} \times \frac{5}{1} = \frac{110}{3}$$

The resultant acceleration is $36\frac{2}{3} \text{ m/s}^2$.

Lesson Summary

When multiplying an expression by negative one, remember to multiply the **entire expression** by negative one.

To multiply fractions, multiply the numerators and multiply the denominators: $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

The **multiplicative properties** are:

- **Commutative Property:** The product of two numbers is the same whichever order the items to be multiplied are written. **Example:** $2 \cdot 3 = 3 \cdot 2$
- **Associative Property:** When three or more numbers are multiplied, the sum is the same regardless of how they are grouped. **Example:** $2 \cdot (3 \cdot 4) = (2 \cdot 3) \cdot 4$
- **Multiplicative Identity Property:** The product of any number and one is the original number. **Example:** $2 \cdot 1 = 2$
- **Distributive property:** The multiplication of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number. **Example:** $4(2 + 3) = 4(2) + 4(3)$

The **multiplicative inverse** of a number is the number which produces 1 when multiplied by the original number. The multiplicative inverse of x is the reciprocal $\frac{1}{x}$. To find the multiplicative inverse of a fraction, simply **invert the fraction:** $\frac{a}{b}$ inverts to $\frac{b}{a}$.

To divide fractions, invert the divisor and multiply: $\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$.

Further Practice

For more practice multiplying fractions, try playing the fraction game at <http://www.aaamath.com/fra66mx2.htm>, or the one at http://www.mathplayground.com/fractions_mult.html. For more practice dividing fractions, try the game at <http://www.aaamath.com/div66ox2.htm> or the one at http://www.mathplayground.com/fractions_div.html.

Review Questions

1. Multiply the following expressions by negative one.

- (a) 25
 (b) -105
 (c) x^2
 (d) $(3 + x)$
 (e) $(3 - x)$
2. Multiply the following rational numbers. Write your answer in the **simplest form**.
- (a) $\frac{5}{12} \times \frac{9}{10}$
 (b) $\frac{2}{3} \times \frac{1}{4}$
 (c) $\frac{3}{4} \times \frac{1}{3}$
 (d) $\frac{15}{11} \times \frac{9}{7}$
 (e) $\frac{1}{13} \times \frac{1}{11}$
 (f) $\frac{7}{27} \times \frac{9}{14}$
 (g) $\left(\frac{3}{5}\right)^2$
 (h) $\frac{1}{11} \times \frac{22}{21} \times \frac{7}{10}$
 (i) $\frac{12}{15} \times \frac{32}{13} \times \frac{10}{2} \times \frac{26}{36}$
3. Find the multiplicative inverse of each of the following.
- (a) 100
 (b) $\frac{2}{8}$
 (c) $-\frac{19}{21}$
 (d) 7
 (e) $-\frac{z^3}{2xy^2}$
4. Divide the following rational numbers. Write your answer in the simplest form.
- (a) $\frac{5}{2} \div \frac{1}{4}$
 (b) $\frac{1}{2} \div \frac{7}{9}$
 (c) $\frac{5}{11} \div \frac{6}{7}$
 (d) $\frac{1}{2} \div \frac{1}{2}$
 (e) $-\frac{x}{2} \div \frac{5}{7}$
 (f) $\frac{1}{2} \div \frac{x}{4y}$
 (g) $\left(-\frac{1}{3}\right) \div \left(-\frac{3}{5}\right)$
 (h) $\frac{7}{2} \div \frac{7}{4}$
 (i) $11 \div \frac{-x}{4}$
5. The label on a can of paint says that it will cover 50 square feet per pint. If I buy a $\frac{1}{8}$ pint sample, it will cover a square two feet long by three feet high. Is the coverage I get more, less or the same as that stated on the label?
6. The world's largest trench digger, "Bagger 288", moves at $\frac{3}{8}$ mph. How long will it take to dig a trench $\frac{2}{3}$ mile long?
7. A $\frac{2}{7}$ Newton force applied to a body of unknown mass produces an acceleration of $\frac{3}{10} m/s^2$. Calculate the mass of the body.

2.4 The Distributive Property

Learning Objectives

- Apply the distributive property.
- Identify parts of an expression.
- Solve real-world problems using the distributive property.

Introduction

At the end of the school year, an elementary school teacher makes a little gift bag for each of his students. Each bag contains one class photograph, two party favors and five pieces of candy. The teacher will distribute the bags among his 28 students. How many of each item does the teacher need?

Apply the Distributive Property

When we have a problem like the one posed in the introduction, **The Distributive Property** can help us solve it. First, we can write an expression for the contents of each bag: Items = (photo + 2 favors + 5 candies), or simply $I = (p + 2f + 5c)$.

For all 28 students, the teacher will need 28 times that number of items, so $I = 28(p + 2f + 5c)$.

Next, **the Distributive Property of Multiplication** tells us that when we have a single term multiplied by a sum of several terms, we can rewrite it by multiplying the single term by each of the other terms separately. In other words, $28(p + 2f + 5c) = 28(p) + 28(2f) + 28(5c)$, which simplifies to $28p + 56f + 140c$. So the teacher needs 28 class photos, 56 party favors and 140 pieces of candy.

You can see why the Distributive Property works by looking at a simple problem where we just have numbers inside the parentheses, and considering the **Order of Operations**.

Example 1

Determine the value of $11(2 - 6)$ using both the Order of Operations and the Distributive Property.

Solution

Order of Operations tells us to evaluate the amount inside the parentheses first:

$$11(2 - 6) = 11(-4) = -44$$

Now let's try it with the Distributive Property:

$$11(2 - 6) = 11(2) - 11(6) = 22 - 66 = -44$$

Note: When applying the Distributive Property you **MUST** take note of any **negative signs!**

Example 2

Use the Distributive Property to determine the following.

a) $11(2x + 6)$

b) $7(3x - 5)$

c) $\frac{2}{7}(3y^2 - 11)$

d) $\frac{2x}{7}(3y^2 - \frac{11}{xy})$

Solution

a) $11(2x + 6) = 11(2x) + 11(6) = 22x + 66$

b) Note the negative sign on the second term.

$$7(3x - 5) = 21x - 35$$

c) $\frac{2}{7}(3y^2 - 11) = \frac{2}{7}(3y^2) + \frac{2}{7}(-11) = \frac{6y^2}{7} - \frac{22}{7}$, or $\frac{6y^2 - 22}{7}$

$$d) \frac{2x}{7} \left(3y^2 - \frac{11}{xy} \right) = \frac{2x}{7} (3y^2) + \frac{2x}{7} \left(-\frac{11}{xy} \right) = \frac{6xy^2}{7} - \frac{22x}{7xy}$$

We can simplify this answer by canceling the x 's in the second fraction, so we end up with $\frac{6xy^2}{7} - \frac{22}{7y}$.

Identify Expressions That Involve the Distributive Property

The Distributive Property can also appear in expressions that don't include parentheses. In Lesson 1.2, we saw how the fraction bar also acts as a grouping symbol. Now we'll see how to use the Distributive Property with fractions.

Example 3

Simplify the following expressions.

a) $\frac{2x+8}{4}$

b) $\frac{9y-2}{3}$

c) $\frac{z+6}{2}$

Solution

Even though these expressions aren't written in a form we usually associate with the Distributive Property, remember that we treat the numerator of a fraction as if it were in parentheses, and that means we can use the Distributive Property here too.

a) $\frac{2x+8}{4}$ can be re-written as $\frac{1}{4}(2x + 8)$. Then we can distribute the $\frac{1}{4}$:

$$\frac{1}{4}(2x + 8) = \frac{2x}{4} + \frac{8}{4} = \frac{x}{2} + 2$$

b) $\frac{9y-2}{3}$ can be re-written as $\frac{1}{3}(9y - 2)$, and then we can distribute the $\frac{1}{3}$:

$$\frac{1}{3}(9y - 2) = \frac{9y}{3} - \frac{2}{3} = 3y - \frac{2}{3}$$

c) Rewrite $\frac{z+6}{2}$ as $\frac{1}{2}(z + 6)$, and distribute the $\frac{1}{2}$:

$$\frac{1}{2}(z + 6) = \frac{z}{2} + \frac{6}{2} = \frac{z}{2} + 3$$

Solve Real-World Problems Using the Distributive Property

The Distributive Property is one of the most common mathematical properties used in everyday life. Any time we have two or more groups of objects, the Distributive Property can help us solve for an unknown.

Example 4

Each student on a field trip into a forest is to be given an emergency survival kit. The kit is to contain a flashlight, a first aid kit, and emergency food rations. Flashlights cost \$12 each, first aid kits are \$7 each and emergency food rations cost \$2 per day. There is \$500 available for the kits and 17 students to provide for. How many days worth of rations can be provided with each kit?

The unknown quantity in this problem is the number of days' rations. This will be x in our expression.

Each kit will contain **one** \$12 flashlight, **one** \$7 first aid kit, and x times \$2 worth of rations, for a total cost of $(12 + 7 + 2x)$ dollars. With 17 kits, therefore, the total cost will be $17(12 + 7 + 2x)$ dollars.

We can use the Distributive Property on this expression:

$$17(12 + 7 + 2x) = 204 + 119 + 34x$$

Since the total cost can be at most \$500, we set the expression equal to 500 and solve for x . (You'll learn in more detail how to solve equations like this in the next chapter.)

$$204 + 119 + 34x = 500$$

$$323 + 34x = 500$$

$$323 + 34x - 323 = 500 - 323$$

$$34x = 177$$

$$\frac{34x}{34} = \frac{177}{34}$$

$$x \approx 5.206$$

Since this represents the number of days' worth of rations that can be bought, we must **round to the next lowest whole number**. We wouldn't have enough money to buy a sixth day of supplies.

Solution

Five days worth of emergency rations can be purchased for each survival kit.

Lesson Summary

- **Distributive Property** The product of a number and the sum of two numbers is equal to the first number times the second number plus the first number times the third number.
- When applying the Distributive Property you **MUST** take note of any **negative signs!**

Further Practice

For more practice using the Distributive Property, try playing the Battleship game at <http://www.quia.com/ba/15357.html>.

Review Questions

1. Use the Distributive Property to simplify the following expressions.
 - (a) $(x + 4) - 2(x + 5)$
 - (b) $\frac{1}{2}(4z + 6)$
 - (c) $(4 + 5) - (5 + 2)$
 - (d) $x(x + 7)$
 - (e) $y(x + 7)$
 - (f) $13x(3y + z)$
 - (g) $x\left(\frac{3}{x} + 5\right)$
 - (h) $xy\left(\frac{1}{x} + \frac{2}{y}\right)$
2. Use the Distributive Property to remove the parentheses from the following expressions.
 - (a) $\frac{1}{2}(x - y) - 4$
 - (b) $0.6(0.2x + 0.7)$
 - (c) $6 + (x - 5) + 7$

- (d) $6 - (x - 5) + 7$
 (e) $4(m + 7) - 6(4 - m)$
 (f) $-5(y - 11) + 2y$
 (g) $-(x - 3y) + \frac{1}{2}(z + 4)$
 (h) $\frac{a}{b} \left(\frac{2}{a} + \frac{3}{b} + \frac{b}{5} \right)$

3. Use the Distributive Property to simplify the following fractions.

- (a) $\frac{8x+12}{4}$
 (b) $\frac{9x+12}{3}$
 (c) $\frac{11x+12}{2}$
 (d) $\frac{3y+2}{6}$
 (e) $-\frac{6z-2}{3}$
 (f) $\frac{7-6p}{3}$
 (g) $\frac{3d-4}{6d}$
 (h) $\frac{12g+8h}{4gh}$

4. A bookcase has five shelves, and each shelf contains seven poetry books and eleven novels. How many of each type of book does the bookcase contain?
5. Amar is making giant holiday cookies for his friends at school. He makes each cookie with 6 oz of cookie dough and decorates them with macadamia nuts. If Amar has 5 lbs of cookie dough ($1 \text{ lb} = 16 \text{ oz}$) and 60 macadamia nuts, calculate the following.
- (a) How many (full) cookies he can make?
 (b) How many macadamia nuts he can put on each cookie, if each is to be identical?
 (c) If 4 cups of flour and 1 cup of sugar went into each pound of cookie dough, how much of each did Amar use to make the 5 pounds of dough?

2.5 Square Roots and Real Numbers

Learning Objectives

- Find square roots.
- Approximate square roots.
- Identify irrational numbers.
- Classify real numbers.
- Graph and order real numbers.

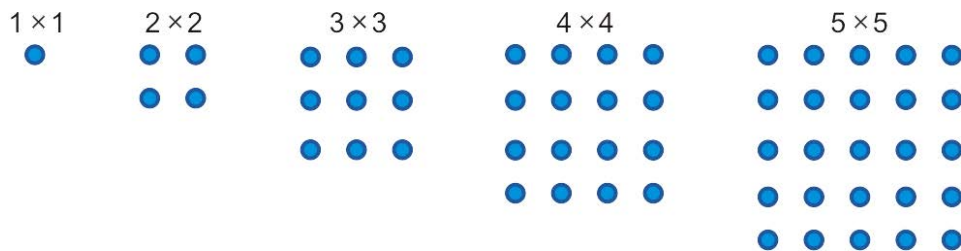
Find Square Roots

The **square root** of a number is a number which, when multiplied by itself, gives the original number. In other words, if $a = b^2$, we say that b is the square root of a .

Note: Negative numbers and positive numbers both yield positive numbers when squared, so each positive number has both a positive and a negative square root. (For example, 3 and -3 can both be squared to yield 9.) The positive square root of a number is called the **principal square root**.

The square root of a number x is written as \sqrt{x} or sometimes as $\sqrt[2]{x}$. The symbol $\sqrt{\quad}$ is sometimes called a **radical sign**.

Numbers with whole-number square roots are called **perfect squares**. The first five perfect squares (1, 4, 9, 16, and 25) are shown below.



You can determine whether a number is a perfect square by looking at its prime factors. If every number in the factor tree appears an even number of times, the number is a perfect square. To find the square root of that number, simply take one of each pair of matching factors and multiply them together.

Example 1

Find the principal square root of each of these perfect squares.

- a) 121
- b) 225
- c) 324
- d) 576

Solution

- a) $121 = 11 \times 11$, so $\sqrt{121} = 11$.
- b) $225 = (5 \times 5) \times (3 \times 3)$, so $\sqrt{225} = 5 \times 3 = 15$.
- c) $324 = (2 \times 2) \times (3 \times 3) \times (3 \times 3)$, so $\sqrt{324} = 2 \times 3 \times 3 = 18$.
- d) $576 = (2 \times 2) \times (2 \times 2) \times (2 \times 2) \times (3 \times 3)$, so $\sqrt{576} = 2 \times 2 \times 2 \times 3 = 24$.

For more practice matching numbers with their square roots, try the Flash games at <http://www.quia.com/jg/65631.html>.

When the prime factors don't pair up neatly, we "factor out" the ones that do pair up and leave the rest under a radical sign. We write the answer as $a\sqrt{b}$, where a is the product of half the paired factors we pulled out and b is the product of the leftover factors.

Example 2

Find the principal square root of the following numbers.

- a) 8
- b) 48
- c) 75
- d) 216

Solution

- a) $8 = 2 \times 2 \times 2$. This gives us one pair of 2's and one leftover 2, so $\sqrt{8} = 2\sqrt{2}$.
- b) $48 = (2 \times 2) \times (2 \times 2) \times 3$, so $\sqrt{48} = 2 \times 2 \times \sqrt{3}$, or $4\sqrt{3}$.
- c) $75 = (5 \times 5) \times 3$, so $\sqrt{75} = 5\sqrt{3}$.
- d) $216 = (2 \times 2) \times 2 \times (3 \times 3) \times 3$, so $\sqrt{216} = 2 \times 3 \times \sqrt{2 \times 3}$, or $6\sqrt{6}$.

Note that in the last example we collected the paired factors first, **then** we collected the unpaired ones under a single radical symbol. Here are the four rules that govern how we treat square roots.

- $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
- $A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab}$
- $\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$
- $\frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}}$

Example 3

Simplify the following square root problems

- $\sqrt{8} \times \sqrt{2}$
- $3\sqrt{4} \times 4\sqrt{3}$
- $\sqrt{12} \div \sqrt{3}$
- $12\sqrt{10} \div 6\sqrt{5}$

Solution

- $\sqrt{8} \times \sqrt{2} = \sqrt{16} = 4$
- $3\sqrt{4} \times 4\sqrt{3} = 12\sqrt{12} = 12\sqrt{(2 \times 2) \times 3} = 12 \times 2\sqrt{3} = 24\sqrt{3}$
- $\sqrt{12} \div \sqrt{3} = \sqrt{\frac{12}{3}} = \sqrt{4} = 2$
- $12\sqrt{10} \div 6\sqrt{5} = \frac{12}{6}\sqrt{\frac{10}{5}} = 2\sqrt{2}$

Approximate Square Roots

Terms like $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{7}$ (square roots of prime numbers) cannot be written as **rational numbers**. That is to say, they cannot be expressed as the ratio of two integers. We call them **irrational numbers**. In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.

To find approximate values for square roots, we use the $\sqrt{\quad}$ or \sqrt{x} button on a calculator. When the number we plug in is a perfect square, or the square of a rational number, we will get an exact answer. When the number is a non-perfect square, the answer will be irrational and will look like a random string of digits. Since the calculator can only show some of the infinitely many digits that are actually in the answer, it is really showing us an **approximate answer**—not exactly the right answer, but as close as it can get.

Example 4

Use a calculator to find the following square roots. Round your answer to three decimal places.

- $\sqrt{99}$
- $\sqrt{5}$
- $\sqrt{0.5}$
- $\sqrt{1.75}$

Solution

- ≈ 9.950
- ≈ 2.236
- ≈ 0.707
- ≈ 1.323

You can also work out square roots by hand using a method similar to long division. (See the web page at <http://www.homeschoolmath.net/teaching/square-root-algorithm.php> for an explanation of this method.)

Identify Irrational Numbers

Not all square roots are irrational, but any square root that can't be reduced to a form with no radical signs in it is irrational. For example, $\sqrt{49}$ is rational because it equals 7, but $\sqrt{50}$ can't be reduced farther than $5\sqrt{2}$. That factor of $\sqrt{2}$ is irrational, making the whole expression irrational.

Example 5

Identify which of the following are rational numbers and which are irrational numbers.

- a) 23.7
- b) 2.8956
- c) π
- d) $\sqrt{6}$
- e) $3.\overline{27}$

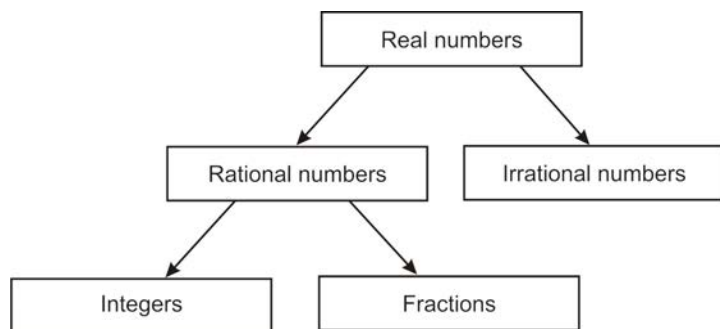
Solution

- a) 23.7 can be written as $23\frac{7}{10}$, so it is rational.
- b) 2.8956 can be written as $2\frac{8956}{10000}$, so it is rational.
- c) $\pi = 3.141592654\dots$ We know from the definition of π that the decimals do not terminate or repeat, so π is irrational.
- d) $\sqrt{6} = \sqrt{2} \times \sqrt{3}$. We can't reduce it to a form without radicals in it, so it is irrational.
- e) $3.\overline{27} = 3.272727272727\dots$ This decimal goes on forever, but it's not random; it repeats in a predictable pattern. Repeating decimals are always rational; this one can actually be expressed as $\frac{36}{11}$.

You can see from this example that any number whose decimal representation has a finite number of digits is rational, since each decimal place can be expressed as a fraction. For example, 0.439 can be expressed as $\frac{4}{10} + \frac{3}{100} + \frac{9}{1000}$, or just $\frac{439}{1000}$. Also, any decimal that repeats is rational, and can be expressed as a fraction. For example, $0.25\overline{38}$ can be expressed as $\frac{25}{100} + \frac{38}{9900}$, which is equivalent to $\frac{2513}{9900}$.

Classify Real Numbers

We can now see how real numbers fall into one of several categories.



If a real number can be expressed as a rational number, it falls into one of two categories. If the denominator

of its **simplest form** is one, then it is an **integer**. If not, it is a fraction (this term also includes decimals, since they can be written as fractions.)

If the number cannot be expressed as the ratio of two integers (i.e. as a fraction), it is **irrational**.

Example 6

Classify the following real numbers.

a) 0

b) -1

c) $\frac{\pi}{3}$

d) $\frac{\sqrt{2}}{3}$

e) $\frac{\sqrt{36}}{9}$

Solution

a) Integer

b) Integer

c) Irrational (Although it's written as a fraction, π is irrational, so any fraction with π in it is also irrational.)

d) Irrational

e) Rational (It simplifies to $\frac{6}{9}$, or $\frac{2}{3}$.)

Lesson Summary

- The **square root** of a number is a number which gives the original number when multiplied by itself. In algebraic terms, for two numbers a and b , if $a = b^2$, then $b = \sqrt{a}$.
- A square root can have two possible values: a positive value called the **principal square root**, and a negative value (the opposite of the positive value).
- A **perfect square** is a number whose square root is an integer.
- Some mathematical properties of square roots are:

$$\begin{aligned} & - \sqrt{a} \times \sqrt{b} = \sqrt{ab} \\ & - A\sqrt{a} \times B\sqrt{b} = AB\sqrt{ab} \\ & - \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} \\ & - \frac{A\sqrt{a}}{B\sqrt{b}} = \frac{A}{B}\sqrt{\frac{a}{b}} \end{aligned}$$

- Square roots of numbers that are not perfect squares (or ratios of perfect squares) are **irrational numbers**. They cannot be written as rational numbers (the ratio of two integers). In decimal form, they have an unending, seemingly random, string of numbers after the decimal point.
- Computing a square root on a calculator will produce an **approximate solution** since the calculator only shows a finite number of digits after the decimal point.

Review Questions

1. Find the following square roots **exactly without using a calculator**, giving your answer in the simplest form.

(a) $\sqrt{25}$

(b) $\sqrt{24}$

- (c) $\sqrt{20}$
- (d) $\sqrt{200}$
- (e) $\sqrt{2000}$
- (f) $\sqrt{\frac{1}{4}}$ (Hint: The division rules you learned can be applied backwards!)
- (g) $\sqrt{\frac{9}{4}}$
- (h) $\sqrt{0.16}$
- (i) $\sqrt{0.1}$
- (j) $\sqrt{0.01}$

2. Use a calculator to find the following square roots. Round to two decimal places.

- (a) $\sqrt{13}$
- (b) $\sqrt{99}$
- (c) $\sqrt{123}$
- (d) $\sqrt{2}$
- (e) $\sqrt{2000}$
- (f) $\sqrt{.25}$
- (g) $\sqrt{1.35}$
- (h) $\sqrt{0.37}$
- (i) $\sqrt{0.7}$
- (j) $\sqrt{0.01}$

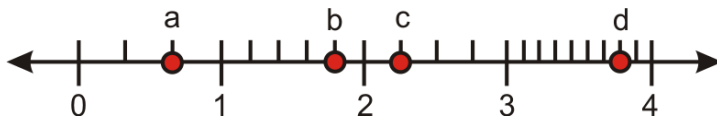
3. Classify the following numbers as an integer, a rational number or an irrational number.

- (a) $\sqrt{0.25}$
- (b) $\sqrt{1.35}$
- (c) $\sqrt{20}$
- (d) $\sqrt{25}$
- (e) $\sqrt{100}$

4. Place the following numbers in numerical order, from lowest to highest.

$$\frac{\sqrt{6}}{2} \quad \frac{61}{50} \quad \sqrt{1.5} \quad \frac{16}{13}$$

5. Use the marked points on the number line and identify each proper fraction.



2.6 Problem-Solving Strategies: Guess and Check, Work Backward

Learning Objectives

- Read and understand given problem situations.
- Develop and use the strategy “Guess and Check.”
- Develop and use the strategy “Work Backward.”
- Plan and compare alternative approaches to solving problems.
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this section, you will learn about the methods of **Guess and Check** and **Working Backwards**. These are very powerful strategies in problem solving and probably the most commonly used in everyday life. Let's review our problem-solving plan.

Step 1

Understand the problem.

Read the problem carefully. Then list all the components and data involved, and assign your variables.

Step 2

Devise a plan – Translate

Come up with a way to solve the problem. Set up an equation, draw a diagram, make a chart or construct a table.

Step 3

Carry out the plan – Solve

This is where you solve the equation you came up with in Step 2.

Step 4

Look – Check and Interpret

Check that the answer makes sense.

Let's now look at some strategies we can use as part of this plan.

Develop and Use the Strategy “Guess and Check”

The strategy for the method “Guess and Check” is to guess a solution and then plug the guess back into the problem to see if you get the correct answer. If the answer is too big or too small, make another guess that will get you closer to the goal, and continue guessing until you arrive at the correct solution. The process might sound long, but often you will find patterns that you can use to make better guesses along the way.

Here is an example of how this strategy is used in practice.

Example 1

Nadia takes a ribbon that is 48 inches long and cuts it in two pieces. One piece is three times as long as the other. How long is each piece?

Solution

Step 1: Understand

We need to find two numbers that add up to 48. One number is three times the other number.

Step 2: Strategy

We guess two random numbers, one three times bigger than the other, and find the sum.

If the sum is too small we guess larger numbers, and if the sum is too large we guess smaller numbers.

Then, we see if any patterns develop from our guesses.

Step 3: Apply Strategy/Solve

Guess	5 and 15	$5 + 15 = 20$	sum is too small
Guess	6 and 18	$6 + 18 = 24$	sum is too small

Our second guess gives us a sum that is exactly half of 48. What if we double that guess?

$$12 + 36 = 48$$

There's our answer. The pieces are 12 and 36 inches long.

Step 4: Check

$12 + 36 = 48$	The pieces add up to 48 inches.
$36 = 3(12)$	One piece is three times as long as the other.

The answer checks out.

Develop and Use the Strategy “Work Backward”

The “Work Backward” method works well for problems where a series of operations is done on an unknown number and you’re only given the result. To use this method, start with the result and apply the operations in reverse order until you find the starting number.

Example 2

Anne has a certain amount of money in her bank account on Friday morning. During the day she writes a check for \$24.50, makes an ATM withdrawal of \$80 and deposits a check for \$235. At the end of the day she sees that her balance is \$451.25. How much money did she have in the bank at the beginning of the day?

Step 1: Understand

We need to find the money in Anne’s bank account at the beginning of the day on Friday.

She took out \$24.50 and \$80 and put in \$235.

She ended up with \$451.25 at the end of the day.

Step 2: Strategy

We start with an unknown amount, do some operations, and end up with a known amount.

We need to start with the result and apply the operations in reverse.

Step 3: Apply Strategy/Solve

Start with \$451.25. Subtract \$235, add \$80, and then add \$24.50.

$$451.25 - 235 + 80 + 24.50 = 320.75$$

Anne had \$320.75 in her account at the beginning of the day on Friday.

Step 4: Check

Anne starts with	\$320.75
She writes a check for \$24.50.	$\$320.75 - 24.50 = \296.25
She withdraws \$80.	$\$296.25 - 80 = \216.25
She deposits \$235.	$\$216.25 + 235 = \451.25

The answer checks out.

Plan and Compare Alternative Approaches to Solving Problems

Most word problems can be solved in more than one way. Often one method is more straightforward than others, but which method is best can depend on what kind of problem you are facing.

Example 3

Nadia's father is 36. He is 16 years older than four times Nadia's age. How old is Nadia?

Solution

This problem can be solved with either of the strategies you learned in this section. Let's solve it using both strategies.

Guess and Check Method

Step 1: Understand

We need to find Nadia's age.

We know that her father is 16 years older than four times her age, or $4 \times (\text{Nadia's age}) + 16$.

We know her father is 36 years old.

Step 2: Strategy

We guess a random number for Nadia's age.

We multiply the number by 4 and add 16 and check to see if the result equals 36.

If the answer is too small, we guess a larger number, and if the answer is too big, we guess a smaller number.

We keep guessing until we get the answer to be 36.

Step 3: Apply strategy/Solve

Guess Nadia's age	10	$4(10) + 16 = 56$	too big for her father's age
Guess a smaller number	9	$4(9) + 16 = 52$	still too big

Guessing 9 for Nadia's age gave us a number that is 16 years too great to be her father's age. But notice that when we decreased Nadia's age by one, her father's age decreased by four. That suggests that we can decrease our final answer by 16 years if we decrease our guess by 4 years.

4 years less than 9 is 5. $4(5) + 16 = 36$, which is the right age.

Answer: Nadia is 5 years old.

Step 4: Check

Nadia is 5 years old. Her father's age is $4(5) + 16 = 36$. This is correct. **The answer checks out.**

Work Backward Method

Step 1: Understand

We need to find Nadia's age.

We know her father is 16 years older than four times her age, or $4 \times (\text{Nadia's age}) + 16$.

We know her father is 36 years old.

Step 2: Strategy

To get from Nadia's age to her father's age, we multiply Nadia's age by four and add 16.

Working backwards means we start with the father's age, subtract 16 and divide by 4.

Step 3: Apply Strategy/Solve

Start with the father's age	36
Subtract 16	$36 - 16 = 20$
Divide by 4	$20 \div 4 = 5$

Answer Nadia is 5 years old.

Step 4: Check

Nadia is 5 years old. Her father's age is $4(5) + 16 = 36$. This is correct. **The answer checks out.**

You see that in this problem, the "Work Backward" strategy is more straightforward than the Guess and Check method. The Work Backward method always works best when we know the result of a series of operations, but not the starting number. In the next chapter, you will learn algebra methods based on the Work Backward method.

Lesson Summary

The four steps of the **problem solving plan** are:

- Understand the problem
- Devise a plan – Translate
- Carry out the plan – Solve
- Look – Check and Interpret

Two common problem solving strategies are:

Guess and Check

Guess a solution and use the guess in the problem to see if you get the correct answer. If the answer is too big or too small, then make another guess that will get you closer to the goal.

Work Backward

This method works well for problems in which a series of operations is applied to an unknown quantity and you are given the resulting number. Start with the result and apply the operations in reverse order until you find the unknown.

Review Questions

1. Finish the problem we started in Example 1.
2. Nadia is at home and Peter is at school which is 6 miles away from home. They start traveling towards each other at the same time. Nadia is walking at 3.5 miles per hour and Peter is skateboarding at 6 miles per hour. When will they meet and how far from home is their meeting place?
3. Peter bought several notebooks at Staples for \$2.25 each; then he bought a few more notebooks at Rite-Aid for \$2 each. He spent the same amount of money in both places and he bought 17 notebooks in all. How many notebooks did Peter buy in each store?
4. Andrew took a handful of change out of his pocket and noticed that he was only holding dimes and quarters in his hand. He counted and found that he had 22 coins that amounted to \$4. How many quarters and how many dimes does Andrew have?
5. Anne wants to put a fence around her rose bed that is one and a half times as long as it is wide. She uses 50 feet of fencing. What are the dimensions of the garden?

6. Peter is outside looking at the pigs and chickens in the yard. Nadia is indoors and cannot see the animals. Peter gives her a puzzle. He tells her that he can see 13 heads and 36 feet and asks her how many pigs and how many chickens are in the yard. Help Nadia find the answer.
7. Andrew invests \$8000 in two types of accounts: a savings account that pays 5.25% interest per year and a more risky account that pays 9% interest per year. At the end of the year he has \$450 in interest from the two accounts. Find the amount of money invested in each account.
8. 450 tickets are sold for a concert: balcony seats for \$35 each and orchestra seats for \$25 each. If the total box office take is \$13,000, how many of each kind of ticket were sold?
9. There is a bowl of candy sitting on our kitchen table. One morning Nadia takes one-sixth of the candy. Later that morning Peter takes one-fourth of the candy that's left. That afternoon, Andrew takes one-fifth of what's left in the bowl and finally Anne takes one-third of what is left in the bowl. If there are 16 candies left in the bowl at the end of the day, how much candy was there at the beginning of the day?
10. Nadia can completely mow the lawn by herself in 30 minutes. Peter can completely mow the lawn by himself in 45 minutes. How long does it take both of them to mow the lawn together?
11. Three monkeys spend a day gathering coconuts together. When they have finished, they are very tired and fall asleep. The following morning, the first monkey wakes up. Not wishing to disturb his friends, he decides to divide the coconuts into three equal piles. There is one left over, so he throws this odd one away, helps himself to his share, and goes home. A few minutes later, the second monkey awakes. Not realizing that the first has already gone, he too divides the coconuts into three equal heaps. He finds one left over, throws the odd one away, helps himself to his fair share, and goes home. In the morning, the third monkey wakes to find that he is alone. He spots the two discarded coconuts, and puts them with the pile, giving him a total of twelve coconuts.
 - (a) How many coconuts did the first two monkeys take?
 - (b) How many coconuts did the monkeys gather in all?
12. Two prime numbers have a product of 51. What are the numbers?
13. Two prime numbers have a product of 65. What are the numbers?
14. The square of a certain positive number is eight more than twice the number. What is the number?
15. Is 91 prime? (Hint: if it's *not* prime, what are its prime factors?)
16. Is 73 prime?
17. Alison's school day starts at 8:30, but today Alison wants to arrive ten minutes early to discuss an assignment with her English teacher. If she is also giving her friend Sherice a ride to school, and it takes her 12 minutes to get to Sherice's house and another 15 minutes to get to school from there, at what time does Alison need to leave her house?
18. At her retail job, Kelly gets a raise of 10% every six months. After her third raise, she now makes \$13.31 per hour. How much did she make when she first started out?
19. Three years ago, Kevin's little sister Becky had her fifth birthday. If Kevin was eight when Becky was born, how old is he now?
20. A warehouse is full of shipping crates; half of them are headed for Boston and the other half for Philadelphia. A truck arrives to pick up 20 of the Boston-bound crates, and then another truck carries away one third of the Philadelphia-bound crates. An hour later, half of the remaining crates are moved onto the loading dock outside. If there are 40 crates left in the warehouse, how many were there originally?
21. Gerald is a bus driver who takes over from another bus driver one day in the middle of his route. He doesn't pay attention to how many passengers are on the bus when he starts driving, but he does notice that three passengers get off at the next stop, a total of eight more get on at the next three stops, two get on and four get off at the next stop, and at the stop after that, a third of the passengers get off.
 - (a) If there are now 14 passengers on the bus, how many were there when Gerald first took over

- the route?
- (b) If half the passengers who got on while Gerald was driving paid the full adult fare of \$1.50, and the other half were students or seniors who paid a discounted fare of \$1.00, how much cash was in the bus's fare box at the beginning of Gerald's shift if there is now \$73.50 in it?
 - (c) When Gerald took over the route, all the passengers currently on the bus had paid full fare. However, some of the passengers who had previously gotten on and off the bus were students or seniors who had paid the discounted fare. Based on the amount of money that was in the cash box, if 28 passengers had gotten on the bus and gotten off before Gerald arrived (in addition to the passengers who had gotten on and were still there when he arrived), how many of those passengers paid the discounted fare?
 - (d) How much money would currently be in the cash box if all the passengers throughout the day had paid the full fare?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9612>.

Chapter 3

Equations of Lines

3.1 One-Step Equations

Learning Objectives

- Solve an equation using addition.
- Solve an equation using subtraction.
- Solve an equation using multiplication.
- Solve an equation using division.

Introduction

Nadia is buying a new mp3 player. Peter watches her pay for the player with a \$100 bill. She receives \$22.00 in change, and from only this information, Peter works out how much the player cost. How much was the player?

In algebra, we can solve problems like this using an **equation**. An **equation** is an algebraic expression that involves an **equals** sign. If we use the letter x to represent the cost of the mp3 player, we can write the equation $x + 22 = 100$. This tells us that the value of the player **plus** the value of the change received is **equal** to the \$100 that Nadia paid.

Another way we could write the equation would be $x = 100 - 22$. This tells us that the value of the player is **equal** to the total amount of money Nadia paid ($100 - 22$). This equation is mathematically equivalent to the first one, but it is easier to solve.

In this chapter, we will learn how to solve for the variable in a one-variable linear equation. **Linear equations** are equations in which each term is either a constant, or a constant times a single variable (raised to the first power). The term linear comes from the word line, because the graph of a linear equation is always a line.

We'll start with simple problems like the one in the last example.

Solving Equations Using Addition and Subtraction

When we work with an algebraic equation, it's important to remember that the two sides have to stay equal for the equation to stay true. We can change the equation around however we want, but whatever

we do to one side of the equation, we have to do to the other side. In the introduction above, for example, we could get from the first equation to the second equation by subtracting 22 from both sides:

$$\begin{aligned}x + 22 &= 100 \\x + 22 - 22 &= 100 - 22 \\x &= 100 - 22\end{aligned}$$

Similarly, we can add numbers to each side of an equation to help solve for our unknown.

Example 1

Solve $x - 3 = 9$.

Solution

To solve an equation for x , we need to **isolate** x —that is, we need to get it by itself on one side of the equals sign. Right now our x has a 3 subtracted from it. To reverse this, we'll add 3—but we must add 3 to **both sides**.

$$\begin{aligned}x - 3 &= 9 \\x - 3 + 3 &= 9 + 3 \\x + 0 &= 9 + 3 \\x &= 12\end{aligned}$$

Example 2

Solve $z - 9.7 = -1.026$

Solution

It doesn't matter what the variable is—the solving process is the same.

$$\begin{aligned}z - 9.7 &= -1.026 \\z - 9.7 + 9.7 &= -1.026 + 9.7 \\z &= 8.674\end{aligned}$$

Make sure you understand the addition of decimals in this example!

Example 3

Solve $x + \frac{4}{7} = \frac{9}{5}$.

Solution

To isolate x , we need to subtract $\frac{4}{7}$ from both sides.

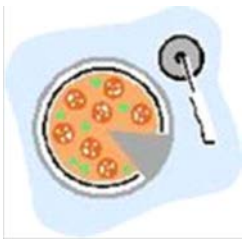
$$\begin{aligned}x + \frac{4}{7} &= \frac{9}{5} \\x + \frac{4}{7} - \frac{4}{7} &= \frac{9}{5} - \frac{4}{7} \\x &= \frac{9}{5} - \frac{4}{7}\end{aligned}$$

Now we have to subtract fractions, which means we need to find the LCD. Since 5 and 7 are both prime, their lowest common multiple is just their product, 35.

$$\begin{aligned}
 x &= \frac{9}{5} - \frac{4}{7} \\
 x &= \frac{7 \cdot 9}{7 \cdot 5} - \frac{4 \cdot 5}{7 \cdot 5} \\
 x &= \frac{63}{35} - \frac{20}{35} \\
 x &= \frac{63 - 20}{35} \\
 x &= \frac{43}{35}
 \end{aligned}$$

Make sure you're comfortable with decimals and fractions! To master algebra, you'll need to work with them frequently.

Solving Equations Using Multiplication and Division



Suppose you are selling pizza for \$1.50 a slice and you can get eight slices out of a single pizza. How much money do you get for a single pizza? It shouldn't take you long to figure out that you get $8 \times \$1.50 = \12.00 . You solved this problem by multiplying. Here's how to do the same thing algebraically, using x to stand for the cost in dollars of the whole pizza.

Example 4

Solve $\frac{1}{8} \cdot x = 1.5$.

Our x is being multiplied by one-eighth. To cancel that out and get x by itself, we have to multiply by the reciprocal, 8. Don't forget to multiply **both sides** of the equation.

$$\begin{aligned}
 8\left(\frac{1}{8} \cdot x\right) &= 8(1.5) \\
 x &= 12
 \end{aligned}$$

Example 5

Solve $\frac{9x}{5} = 5$.

$\frac{9x}{5}$ is equivalent to $\frac{9}{5} \cdot x$, so to cancel out that $\frac{9}{5}$, we multiply by the reciprocal, $\frac{5}{9}$.

$$\begin{aligned}
 \frac{5}{9}\left(\frac{9x}{5}\right) &= \frac{5}{9}(5) \\
 x &= \frac{25}{9}
 \end{aligned}$$

Example 6

Solve $0.25x = 5.25$.

0.25 is the decimal equivalent of one fourth, so to cancel out the 0.25 factor we would multiply by 4.

$$\begin{aligned}4(0.25x) &= 4(5.25) \\x &= 21\end{aligned}$$

Solving by division is another way to isolate x . Suppose you buy five identical candy bars, and you are charged \$3.25. How much did each candy bar cost? You might just divide \$3.25 by 5, but let's see how this problem looks in algebra.

Example 7

Solve $5x = 3.25$.

To cancel the 5, we divide both sides by 5.

$$\begin{aligned}\frac{5x}{5} &= \frac{3.25}{5} \\x &= 0.65\end{aligned}$$

Example 8

Solve $7x = \frac{5}{11}$.

Divide both sides by 7.

$$\begin{aligned}x &= \frac{5}{11.7} \\x &= \frac{5}{77}\end{aligned}$$

Example 9

Solve $1.375x = 1.2$.

Divide by 1.375

$$\begin{aligned}x &= \frac{1.2}{1.375} \\x &= 0.8\overline{72}\end{aligned}$$

Notice the bar above the final two decimals; it means that those digits recur, or repeat. The full answer is $0.872727272727272\dots$

To see more examples of one - and two-step equation solving, watch the Khan Academy video series starting at <http://www.youtube.com/watch?v=bAerID24QJ0>.

Solve Real-World Problems Using Equations

Example 10

In the year 2017, Anne will be 45years old. In what year was Anne born?

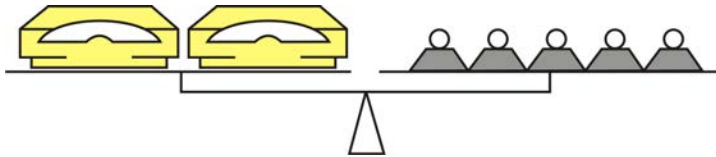
The unknown here is the year Anne was born, so that's our variable x . Here's our equation:

$$\begin{aligned}x + 45 &= 2017 \\x + 45 - 45 &= 2017 - 45 \\x &= 1972\end{aligned}$$

Anne was born in 1972.

Example 11

A mail order electronics company stocks a new mini DVD player and is using a balance to determine the shipping weight. Using only one-pound weights, the shipping department found that the following arrangement balances:



How much does each DVD player weigh?

Solution

Since the system balances, the total weight on each side must be equal. To write our equation, we'll use x for the weight of one DVD player, which is unknown. There are two DVD players, weighing a total of $2x$ pounds, on the left side of the balance, and on the right side are 5 1-pound weights, weighing a total of 5 pounds. So our equation is $2x = 5$. Dividing both sides by 2 gives us $x = 2.5$.

Each DVD player weighs 2.5 pounds.

Example 12

In 2004, Takeru Kobayashi of Nagano, Japan, ate 53.5 hot dogs in 12 minutes. This was 3 more hot dogs than his own previous world record, set in 2002. Calculate:

- How many minutes it took him to eat one hot dog.
- How many hot dogs he ate per minute.
- What his old record was.

Solution

a) We know that the total time for 53.5 hot dogs is 12 minutes. We want to know the time for one hot dog, so that's x . Our equation is $53.5x = 12$. Then we divide both sides by 53.5 to get $x = \frac{12}{53.5}$, or $x = 0.224$ minutes.

We can also multiply by 60 to get the time in seconds; 0.224 minutes is about 13.5 seconds. So that's how long it took Takeru to eat one hot dog.

b) Now we're looking for hot dogs per minute instead of minutes per hot dog. We'll use the variable y instead of x this time so we don't get the two confused. 12 minutes, times the number of hot dogs per minute, equals the total number of hot dogs, so $12y = 53.5$. Dividing both sides by 12 gives us $y = \frac{53.5}{12}$, or $y = 4.458$ hot dogs per minute.

c) We know that his new record is 53.5, and we know that's three more than his old record. If we call his old record z , we can write the following equation: $z + 3 = 53.5$. Subtracting 3 from both sides gives us $z = 50.5$. So Takeru's old record was 50.5 hot dogs in 12 minutes.

Lesson Summary

- An equation in which each term is either a constant or the product of a constant and a single variable is a **linear equation**.
- We can add, subtract, multiply, or divide both sides of an equation by the same value and still have an equivalent equation.

- To solve an equation, **isolate** the unknown variable on one side of the equation by applying one or more arithmetic operations to both sides.

Review Questions

1. Solve the following equations for x .

- (a) $x + 11 = 7$
- (b) $x - 1.1 = 3.2$
- (c) $7x = 21$
- (d) $4x = 1$
- (e) $\frac{5x}{12} = \frac{2}{3}$
- (f) $x + \frac{5}{2} = \frac{1}{3}$
- (g) $x - \frac{5}{6} = \frac{8}{8}$
- (h) $0.01x = 11$

2. Solve the following equations for the unknown variable.

- (a) $q - 13 = -13$
- (b) $z + 1.1 = 3.0001$
- (c) $21s = 3$
- (d) $t + \frac{1}{2} = \frac{1}{3}$
- (e) $\frac{7f}{11} = \frac{7}{11}$
- (f) $\frac{3}{4} = -\frac{1}{2} - y$
- (g) $6r = \frac{3}{8}$
- (h) $\frac{9b}{16} = \frac{8}{8}$

3. Peter is collecting tokens on breakfast cereal packets in order to get a model boat. In eight weeks he has collected 10 tokens. He needs 25 tokens for the boat. Write an equation and determine the following information.

- (a) How many more tokens he needs to collect, n .
- (b) How many tokens he collects per week, w .
- (c) How many more weeks remain until he can send off for his boat, r .

4. Juan has baked a cake and wants to sell it in his bakery. He is going to cut it into 12 slices and sell them individually. He wants to sell it for three times the cost of making it. The ingredients cost him \$8.50, and he allowed \$1.25 to cover the cost of electricity to bake it. Write equations that describe the following statements

- (a) The amount of money that he sells the cake for (u).
- (b) The amount of money he charges for each slice (c).
- (c) The total profit he makes on the cake (w).

5. Jane is baking cookies for a large party. She has a recipe that will make one batch of two dozen cookies, and she decides to make five batches. To make five batches, she finds that she will need 12.5 cups of flour and 15 eggs.

- (a) How many cookies will she make in all?
- (b) How many cups of flour go into one batch?
- (c) How many eggs go into one batch?
- (d) If Jane only has a dozen eggs on hand, how many more does she need to make five batches?
- (e) If she doesn't go out to get more eggs, how many batches can she make? How many cookies will that be?

3.2 Two-Step Equations

Learning Objectives

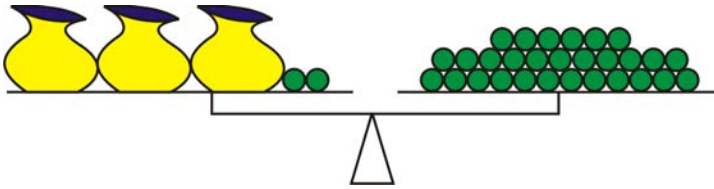
- Solve a two-step equation using addition, subtraction, multiplication, and division.
- Solve a two-step equation by combining like terms.
- Solve real-world problems using two-step equations.

Solve a Two-Step Equation

We've seen how to solve for an unknown by isolating it on one side of an equation and then evaluating the other side. Now we'll see how to solve equations where the variable takes more than one step to isolate.

Example 1

Rebecca has three bags containing the same number of marbles, plus two marbles left over. She places them on one side of a balance. Chris, who has more marbles than Rebecca, adds marbles to the other side of the balance. He finds that with 29 marbles, the scales balance. How many marbles are in each bag? Assume the bags weigh nothing.



Solution

We know that the system balances, so the weights on each side must be equal. If we use x to represent the number of marbles in each bag, then we can see that on the left side of the scale we have three bags (each containing x marbles) plus two extra marbles, and on the right side of the scale we have 29 marbles. The balancing of the scales is similar to the balancing of the following equation.

$$3x + 2 = 29$$

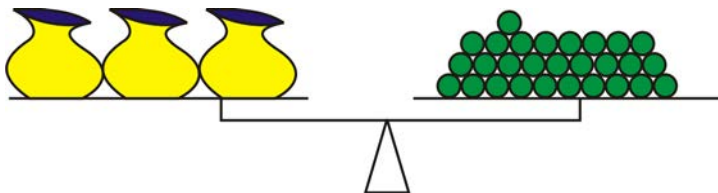
“Three bags plus two marbles **equals** 29 marbles”

To solve for x , we need to first get all the variables (terms containing an x) alone on one side of the equation. We've already got all the x 's on one side; now we just need to isolate them.

$$\begin{aligned} 3x + 2 &= 29 \\ 3x + 2 - 2 &= 29 - 2 && \text{Get rid of the 2 on the left by subtracting it from both sides.} \\ 3x &= 27 \\ \frac{3x}{3} &= \frac{27}{3} && \text{Divide both sides by 3.} \\ x &= 9 \end{aligned}$$

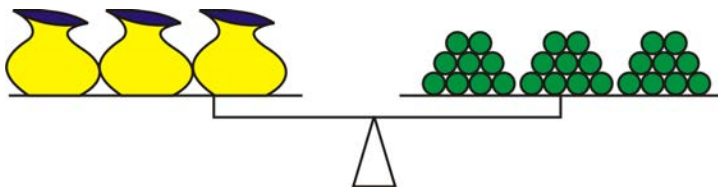
There are nine marbles in each bag.

We can do the same with the real objects as we did with the equation. Just as we subtracted 2 from both sides of the equals sign, we could remove two marbles from each side of the scale. Because we removed the same number of marbles from each side, we know the scales will still balance.



Then, because there are three bags of marbles on the left-hand side of the scale, we can divide the marbles on the right-hand side into three equal piles. You can see that there are nine marbles in each.

Three bags of marbles **balances** three piles of nine marbles.



So each bag of marbles balances nine marbles, meaning that each bag contains nine marbles.

Check out <http://www.mste.uiuc.edu/pavel/java/balance/> for more interactive balance beam activities!

Example 2

Solve $6(x + 4) = 12$.

This equation has the x buried in parentheses. To dig it out, we can proceed in one of two ways: we can either distribute the six on the left, or divide both sides by six to remove it from the left. Since the right-hand side of the equation is a multiple of six, it makes sense to divide. That gives us $x + 4 = 2$. Then we can subtract 4 from both sides to get $x = -2$.

Example 3

Solve $\frac{x-3}{5} = 7$.

It's always a good idea to get rid of fractions first. Multiplying both sides by 5 gives us $x - 3 = 35$, and then we can add 3 to both sides to get $x = 38$.

Example 4

Solve $\frac{5}{9}(x + 1) = \frac{2}{7}$.

First, we'll cancel the fraction on the left by multiplying by the reciprocal (the multiplicative inverse).

$$\begin{aligned} \frac{9}{5} \cdot \frac{5}{9}(x + 1) &= \frac{9}{5} \cdot \frac{2}{7} \\ (x + 1) &= \frac{18}{35} \end{aligned}$$

Then we subtract 1 from both sides. ($\frac{35}{35}$ is equivalent to 1.)

$$\begin{aligned} x + 1 &= \frac{18}{35} \\ x + 1 - 1 &= \frac{18}{35} - \frac{35}{35} \\ x &= \frac{18 - 35}{35} \\ x &= \frac{-17}{35} \end{aligned}$$

These examples are called **two-step equations**, because we need to perform two separate operations on the equation to isolate the variable.

Solve a Two-Step Equation by Combining Like Terms

When we look at a linear equation we see two kinds of terms: those that contain the unknown variable, and those that don't. When we look at an equation that has an x on both sides, we know that in order to solve it, we need to get all the x -terms on one side of the equation. This is called **combining like terms**. The terms with an x in them are **like terms** because they contain the same variable (or, as you will see in later chapters, the same combination of variables).

Table 3.1:

Like Terms	Unlike Terms
$4x, 10x, -3.5x,$ and $\frac{x}{12}$	$3x$ and $3y$
$3y, 0.000001y,$ and y	$4xy$ and $4x$
$xy, 6xy,$ and $2.39xy$	$0.5x$ and 0.5

To add or subtract like terms, we can use the Distributive Property of Multiplication.

$$\begin{aligned}3x + 4x &= (3 + 4)x = 7x \\0.03xy - 0.01xy &= (0.03 - 0.01)xy = 0.02xy \\-y + 16y + 5y &= (-1 + 16 + 5)y = 10y \\5z + 2z - 7z &= (5 + 2 - 7)z = 0z = 0\end{aligned}$$

To solve an equation with two or more like terms, we need to combine the terms first.

Example 5

Solve $(x + 5) - (2x - 3) = 6$.

There are two like terms: the x and the $-2x$ (don't forget that the negative sign applies to everything in the parentheses). So we need to get those terms together. The associative and distributive properties let us rewrite the equation as $x + 5 - 2x + 3 = 6$, and then the commutative property lets us switch around the terms to get $x - 2x + 5 + 3 = 6$, or $(x - 2x) + (5 + 3) = 6$.

$(x - 2x)$ is the same as $(1 - 2)x$, or $-x$, so our equation becomes $-x + 8 = 6$

Subtracting 8 from both sides gives us $-x = -2$.

And finally, multiplying both sides by -1 gives us $x = 2$.

Example 6

Solve $\frac{x}{2} - \frac{x}{3} = 6$.

This problem requires us to deal with fractions. We need to write all the terms on the left over a common denominator of six.

$$\frac{3x}{6} - \frac{2x}{6} = 6$$

Then we subtract the fractions to get $\frac{x}{6} = 6$.

Finally we multiply both sides by 6 to get $x = 36$.

Solve Real-World Problems Using Two-Step Equations

The hardest part of solving word problems is translating from words to an equation. First, you need to look to see what the equation is asking. What is the **unknown** for which you have to solve? That will be what your **variable** stands for. Then, follow what is going on with your variable all the way through the problem.

Example 7



An emergency plumber charges \$65 as a call-out fee plus an additional \$75 per hour. He arrives at a house at 9:30 and works to repair a water tank. If the total repair bill is \$196.25, at what time was the repair completed?

In order to solve this problem, we collect the information from the text and convert it to an equation.

Unknown: time taken in hours – this will be our x

The bill is made up of two parts: a call out fee and a per-hour fee. The call out is a flat fee, and independent of x —it's the same no matter how many hours the plumber works. The per-hour part depends on the number of hours (x). So the total fee is \$65 (no matter what) plus $75x$ (where x is the number of hours), or $65 + 75x$.

Looking at the problem again, we also can see that the total bill is \$196.25. So our final equation is $196.25 = 65 + 75x$.

Solving for x :

$$196.25 = 65 + 75x \quad \text{Subtract 65 from both sides.}$$

$$131.25 = 75x \quad \text{Divide both sides by 75.}$$

$$1.75 = x \quad \text{The job took 1.75 hours.}$$

Solution

The repair job was completed 1.75 hours after 9:30, so it was completed at 11:15AM.

Example 8

When Asia was young her Daddy marked her height on the door frame every month. Asia's Daddy noticed that between the ages of one and three, he could predict her height (in inches) by taking her age in months, adding 75 inches and multiplying the result by one-third. Use this information to answer the following:

- Write an equation linking her predicted height, h , with her age in months, m .*
- Determine her predicted height on her second birthday.*
- Determine at what age she is predicted to reach three feet tall.*

Solution

a) To convert the text to an equation, first determine the type of equation we have. We are going to have an equation that links **two variables**. Our unknown will change, depending on the information we are

given. For example, we could solve for height given age, or solve for age given height. However, the text gives us a way to determine **height**. Our equation will start with “ $h =$ ”.

The text tells us that we can predict her height by taking her age in months, adding 75, and multiplying by $\frac{1}{3}$. So our equation is $h = (m + 75) \cdot \frac{1}{3}$, or $h = \frac{1}{3}(m + 75)$.

b) To predict Asia’s height on her second birthday, we substitute $m = 24$ into our equation (because 2 years is 24 months) and solve for h .

$$\begin{aligned}h &= \frac{1}{3}(24 + 75) \\h &= \frac{1}{3}(99) \\h &= 33\end{aligned}$$

Asia’s height on her second birthday was predicted to be 33 inches.

c) To determine the predicted age when she reached three feet, substitute $h = 36$ into the equation and solve for m .

$$\begin{aligned}36 &= \frac{1}{3}(m + 75) \\108 &= m + 75 \\33 &= m\end{aligned}$$

Asia was predicted to be 33 months old when her height was three feet.

Example 9

To convert temperatures in Fahrenheit to temperatures in Celsius, follow the following steps: Take the temperature in degrees Fahrenheit and subtract 32. Then divide the result by 1.8 and this gives the temperature in degrees Celsius.

a) Write an equation that shows the conversion process.

b) Convert 50 degrees Fahrenheit to degrees Celsius.

c) Convert 25 degrees Celsius to degrees Fahrenheit.

d) Convert -40 degrees Celsius to degrees Fahrenheit.

a) The text gives the process to convert Fahrenheit to Celsius. We can write an equation using two variables. We will use f for temperature in Fahrenheit, and c for temperature in Celsius.

First we take the temperature in Fahrenheit and subtract 32.

$$f - 32$$

Then divide by 1.8.

$$\frac{f - 32}{1.8}$$

This equals the temperature in Celsius.

$$c = \frac{f - 32}{1.8}$$

In order to convert from one temperature scale to another, simply substitute in for whichever temperature you know, and solve for the one you don’t know.

b) To convert 50 degrees Fahrenheit to degrees Celsius, substitute $f = 50$ into the equation.

$$\begin{aligned}c &= \frac{50 - 32}{1.8} \\c &= \frac{18}{1.8} \\c &= 10\end{aligned}$$

50 degrees Fahrenheit is equal to 10 degrees Celsius.

c) To convert 25 degrees Celsius to degrees Fahrenheit, substitute $c = 25$ into the equation:

$$\begin{aligned}25 &= \frac{f - 32}{1.8} \\45 &= f - 32 \\77 &= f\end{aligned}$$

25 degrees Celsius is equal to 77 degrees Fahrenheit.

d) To convert -40 degrees Celsius to degrees Fahrenheit, substitute $c = -40$ into the equation.

$$\begin{aligned}-40 &= \frac{f - 32}{1.8} \\-72 &= f - 32 \\-40 &= f\end{aligned}$$

-40 degrees Celsius is equal to -40 degrees Fahrenheit. (No, that's not a mistake! This is the one temperature where they are equal.)

Lesson Summary

- Some equations require more than one operation to solve. Generally it, is good to go from the outside in. If there are parentheses around an expression with a variable in it, cancel what is outside the parentheses first.
- Terms with the same variable in them (or no variable in them) are **like terms**. **Combine like terms** (adding or subtracting them from each other) to simplify the expression and solve for the unknown.

Review Questions

1. Solve the following equations for the unknown variable.

- $1.3x - 0.7x = 12$
- $6x - 1.3 = 3.2$
- $5x - (3x + 2) = 1$
- $4(x + 3) = 1$
- $5q - 7 = \frac{2}{3}$
- $\frac{3}{5}x + \frac{5}{2} = \frac{2}{3}$
- $s - \frac{3s}{8} = \frac{5}{6}$
- $0.1y + 11 = 0$
- $\frac{5q-7}{12} = \frac{2}{3}$
- $\frac{5(q-7)}{12} = \frac{2}{3}$
- $33t - 99 = 0$
- $5p - 2 = 32$
- $10y + 5 = 10$
- $10(y + 5) = 10$
- $10y + 5y = 10$
- $10(y + 5y) = 10$

2. Jade is stranded downtown with only \$10 to get home. Taxis cost \$0.75 per mile, but there is an additional \$2.35 hire charge. Write a formula and use it to calculate how many miles she can travel with her money.
3. Jasmin's Dad is planning a surprise birthday party for her. He will hire a bouncy castle, and will provide party food for all the guests. The bouncy castle costs \$150 for the afternoon, and the food will cost \$3 per person. Andrew, Jasmin's Dad, has a budget of \$300. Write an equation and use it to determine the maximum number of guests he can invite.
4. The local amusement park sells summer memberships for \$50 each. Normal admission to the park costs \$25; admission for members costs \$15.
 - (a) If Darren wants to spend no more than \$100 on trips to the amusement park this summer, how many visits can he make if he buys a membership with part of that money?
 - (b) How many visits can he make if he does not?
 - (c) If he increases his budget to \$160, how many visits can he make as a member?
 - (d) And how many as a non-member?
5. For an upcoming school field trip, there must be one adult supervisor for every five children.
 - (a) If the bus seats 40 people, how many children can go on the trip?
 - (b) How many children can go if a second 40-person bus is added?
 - (c) Four of the adult chaperones decide to arrive separately by car. Now how many children can go in the two buses?

3.3 Multi-Step Equations

Learning Objectives

- Solve a multi-step equation by combining like terms.
- Solve a multi-step equation using the distributive property.
- Solve real-world problems using multi-step equations.

Solving Multi-Step Equations by Combining Like Terms

We've seen that when we solve for an unknown variable, it can take just one or two steps to get the terms in the right places. Now we'll look at solving equations that take several steps to isolate the unknown variable. Such equations are referred to as **multi-step equations**.

In this section, we'll simply be combining the steps we already know how to do. Our goal is to end up with all the constants on one side of the equation and all the variables on the other side. We'll do this by collecting like terms. Don't forget, like terms have the same combination of variables in them.

Example 1

Solve $\frac{3x+4}{3} - 5x = 6$.

Before we can combine the variable terms, we need to get rid of that fraction.

First let's put all the terms on the left over a common denominator of three: $\frac{3x+4}{3} - \frac{15x}{3} = 6$.

Combining the fractions then gives us $\frac{3x+4-15x}{3} = 6$.

Combining like terms in the numerator gives us $\frac{4-12x}{3} = 6$.

Multiplying both sides by 3 gives us $4 - 12x = 18$.

Subtracting 4 from both sides gives us $-12x = 14$.

And finally, dividing both sides by -12 gives us $x = -\frac{14}{12}$, which reduces to $x = -\frac{7}{6}$.

Solving Multi-Step Equations Using the Distributive Property

You may have noticed that when one side of the equation is multiplied by a constant term, we can either distribute it or just divide it out. If we can divide it out without getting awkward fractions as a result, then that's usually the better choice, because it gives us smaller numbers to work with. But if dividing would result in messy fractions, then it's usually better to distribute the constant and go from there.

Example 2

Solve $7(2x - 5) = 21$.

The first thing we want to do here is get rid of the parentheses. We could use the Distributive Property, but it just so happens that 7 divides evenly into 21. That suggests that dividing both sides by 7 is the easiest way to solve this problem.

If we do that, we get $2x - 5 = \frac{21}{7}$ or just $2x - 5 = 3$. Then all we need to do is add 5 to both sides to get $2x = 8$, and then divide by 2 to get $x = 4$.

Example 3

Solve $17(3x + 4) = 7$.

Once again, we want to get rid of those parentheses. We could divide both sides by 17, but that would give us an inconvenient fraction on the right-hand side. In this case, distributing is the easier way to go.

Distributing the 17 gives us $51x + 68 = 7$. Then we subtract 68 from both sides to get $51x = -61$, and then we divide by 51 to get $x = \frac{-61}{51}$. (Yes, that's a messy fraction too, but since it's our final answer and we don't have to do anything else with it, we don't really care how messy it is.)

Example 4

Solve $4(3x - 4) - 7(2x + 3) = 3$.

Before we can collect like terms, we need to get rid of the parentheses using the Distributive Property. That gives us $12x - 16 - 14x - 21 = 3$, which we can rewrite as $(12x - 14x) + (-16 - 21) = 3$. This in turn simplifies to $-2x - 37 = 3$.

Next we add 37 to both sides to get $-2x = 40$.

And finally, we divide both sides by -2 to get $x = -20$.

Example 5

Solve the following equation for x: $0.1(3.2 + 2x) + \frac{1}{2}\left(3 - \frac{x}{5}\right) = 0$

This function contains both fractions and decimals. We should convert all terms to one or the other. It's often easier to convert decimals to fractions, but in this equation the fractions are easy to convert to decimals—and with decimals we don't need to find a common denominator!

In decimal form, our equation becomes $0.1(3.2 + 2x) + 0.5(3 - 0.2x) = 0$.

Distributing to get rid of the parentheses, we get $0.32 + 0.2x + 1.5 - 0.1x = 0$.

Collecting and combining like terms gives us $0.1x + 1.82 = 0$.

Then we can subtract 1.82 from both sides to get $0.1x = -1.82$, and finally divide by 0.1 (or multiply by 10) to get $x = -18.2$.

Solving Real-World Problems Using Multi-Step Equations

Example 6

A growers' cooperative has a farmer's market in the town center every Saturday. They sell what they have grown and split the money into several categories. 8.5% of all the money taken in is set aside for sales tax. \$150 goes to pay the rent on the space they occupy. What remains is split evenly between the seven growers. How much total money is taken in if each grower receives a \$175 share?

Let's translate the text above into an equation. The unknown is going to be the total money taken in dollars. We'll call this x .

"8.5% of all the money taken in is set aside for sales tax." This means that 91.5% of the money remains. This is $0.915x$.

"\$150 goes to pay the rent on the space they occupy." This means that what's left is $0.915x - 150$.

"What remains is split evenly between the 7 growers." That means each grower gets $\frac{0.915x-150}{7}$.

If each grower's share is \$175, then our equation to find x is $\frac{0.915x-150}{7} = 175$.

First we multiply both sides by 7 to get $0.915x - 150 = 1225$.

Then add 150 to both sides to get $0.915x = 1375$.

Finally divide by 0.915 to get $x \approx 1502.7322$. Since we want our answer in dollars and cents, we round to two decimal places, or \$1502.73.

The workers take in a total of \$1502.73.

Example 7

A factory manager is packing engine components into wooden crates to be shipped on a small truck. The truck is designed to hold sixteen crates, and will safely carry a 1200 lb cargo. Each crate weighs 12 lbs empty. How much weight should the manager instruct the workers to put in each crate in order to get the shipment weight as close as possible to 1200 lbs?

The unknown quantity is the weight to put in each box, so we'll call that x .

Each crate when full will weigh $x + 12$ lbs, so all 16 crates together will weigh $16(x + 12)$ lbs.

We also know that all 16 crates together should weigh 1200 lbs, so we can say that $16(x + 12) = 1200$.

To solve this equation, we can start by dividing both sides by 16: $x + 12 = \frac{1200}{16} = 75$.

Then subtract 12 from both sides: $x = 63$.

The manager should tell the workers to put 63 lbs of components in each crate.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

Example 8

A scientist is trying to deduce the resistance of an unknown component. He labels the resistance of the unknown component $x \Omega$. The resistance of a circuit containing a number of these components is $(5x + 20)\Omega$. If a 120 volt potential difference across the circuit produces a current of 2.5 amps, calculate the resistance of the unknown component.

Solution

To solve this, we need to start with the equation $V = I \cdot R$ and substitute in $V = 120$, $I = 2.5$, and $R = 5x + 20$. That gives us $120 = 2.5(5x + 20)$.

Distribute the 2.5 to get $120 = 12.5x + 50$.

Subtract 50 from both sides to get $70 = 12.5x$.

Finally, divide by 12.5 to get $5.6 = x$.

The unknown components have a resistance of 5.6 Ω .

Distance, Speed and Time

The speed of a body is the distance it travels per unit of time. That means that we can also find out how far an object moves in a certain amount of time if we know its speed: we use the equation “distance = speed \times time.”

Example 8

Shanice’s car is traveling 10 miles per hour slower than twice the speed of Brandon’s car. She covers 93 miles in 1 hour 30 minutes. How fast is Brandon driving?

Solution

Here, we don’t know either Brandon’s speed or Shanice’s, but since the question asks for Brandon’s speed, that’s what we’ll use as our variable x .

The distance Shanice covers in miles is 93, and the time in hours is 1.5. Her speed is 10 less than twice Brandon’s speed, or $2x - 10$ miles per hour. Putting those numbers into the equation gives us $93 = 1.5(2x - 10)$.

First we distribute, to get $93 = 3x - 15$.

Then we add 15 to both sides to get $108 = 3x$.

Finally we divide by 3 to get $36 = x$.

Brandon is driving at 36 miles per hour.

We can check this answer by considering the situation another way: we can solve for Shanice’s speed instead of Brandon’s and then check that against Brandon’s speed. We’ll use y for Shanice’s speed since we already used x for Brandon’s.

The equation for Shanice’s speed is simply $93 = 1.5y$. We can divide both sides by 1.5 to get $62 = y$, so Shanice is traveling at 62 miles per hour.

The problem tells us that Shanice is traveling 10 mph slower than twice Brandon’s speed; that would mean that 62 is equal to 2 times 36 minus 10. Is that true? Well, 2 times 36 is 72, minus 10 is 62. The answer checks out.

In algebra, there’s almost always more than one method of solving a problem. If time allows, it’s always a good idea to try to solve the problem using two different methods just to confirm that you’ve got the answer right.

Speed of Sound

The speed of sound in dry air, v , is given by the equation $v = 331 + 0.6T$, where T is the temperature in Celsius and v is the speed of sound in meters per second.

Example 9

Tashi hits a drainpipe with a hammer and 250 meters away Minh hears the sound and hits his own drainpipe. Unfortunately, there is a one second delay between him hearing the sound and hitting his own pipe. Tashi accurately measures the time between her hitting the pipe and hearing Minh's pipe at 2.46 seconds. What is the temperature of the air?

This is a complex problem and we need to be careful in writing our equations. First of all, the distance the sound travels is equal to the speed of sound multiplied by the time, and the speed is given by the equation above. So the distance equals $(331 + 0.6T) \times \text{time}$, and the time is $2.46 - 1$ (because for 1 second out of the 2.46 seconds measured, there was no sound actually traveling). We also know that the distance is 250×2 (because the sound traveled from Tashi to Minh and back again), so our equation is $250 \times 2 = (331 + 0.6T)(2.46 - 1)$, which simplifies to $500 = 1.46(331 + 0.6T)$.

Distributing gives us $500 = 483.26 + 0.876T$, and subtracting 483.26 from both sides gives us $16.74 = 0.876T$. Then we divide by 0.876 to get $T \approx 19.1$.

The temperature is about 19.1 degrees Celsius.

Lesson Summary

- Multi-step equations are slightly more complex than one - and two-step equations, but use the same basic techniques.
- If dividing a number outside of parentheses will produce fractions, it is often better to use the **Distributive Property** to expand the terms and then combine like terms to solve the equation.

Review Questions

1. Solve the following equations for the unknown variable.

(a) $3(x - 1) - 2(x + 3) = 0$

(b) $3(x + 3) - 2(x - 1) = 0$

(c) $7(w + 20) - w = 5$

(d) $5(w + 20) - 10w = 5$

(e) $9(x - 2) - 3x = 3$

(f) $12(t - 5) + 5 = 0$

(g) $2(2d + 1) = \frac{2}{3}$

(h) $2\left(5a - \frac{1}{3}\right) = \frac{2}{7}$

(i) $\frac{2}{9}\left(i + \frac{2}{3}\right) = \frac{2}{5}$

(j) $4\left(v + \frac{1}{4}\right) = \frac{35}{2}$

(k) $\frac{g}{10} = \frac{6}{3}$

(l) $\frac{s-4}{11} = \frac{2}{5}$

(m) $\frac{2k}{7} = \frac{3}{8}$

(n) $\frac{7x+4}{3} = \frac{9}{2}$

(o) $\frac{9y-3}{6} = \frac{5}{2}$

(p) $\frac{r}{3} + \frac{r}{2} = 7$

(q) $\frac{p}{16} - \frac{2p}{3} = \frac{1}{9}$

(r) $\frac{m+3}{2} - \frac{m}{4} = \frac{1}{3}$

(s) $5\left(\frac{k}{3} + 2\right) = \frac{32}{3}$

(t) $\frac{3}{z} = \frac{2}{5}$

(u) $\frac{2}{r} + 2 = \frac{10}{3}$

(v) $\frac{12}{5} = \frac{3+z}{z}$

- An engineer is building a suspended platform to raise bags of cement. The platform has a mass of 200 kg, and each bag of cement is 40 kg. He is using two steel cables, each capable of holding 250 kg. Write an equation for the number of bags he can put on the platform at once, and solve it.
- A scientist is testing a number of identical components of unknown resistance which he labels $x\Omega$. He connects a circuit with resistance $(3x + 4)\Omega$ to a steady 12 volt supply and finds that this produces a current of 1.2 amps. What is the value of the unknown resistance?
- Lydia inherited a sum of money. She split it into five equal parts. She invested three parts of the money in a high-interest bank account which added 10% to the value. She placed the rest of her inheritance plus \$500 in the stock market but lost 20% on that money. If the two accounts end up with exactly the same amount of money in them, how much did she inherit?
- Pang drove to his mother's house to drop off her new TV. He drove at 50 miles per hour there and back, and spent 10 minutes dropping off the TV. The entire journey took him 94 minutes. How far away does his mother live?

3.4 Equations with Variables on Both Sides

Learning Objectives

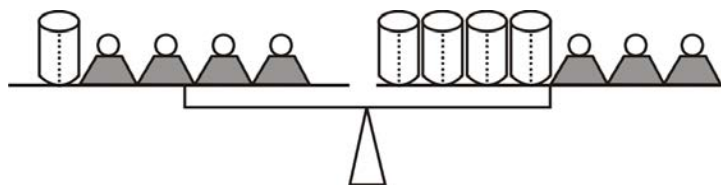
- Solve an equation with variables on both sides.
- Solve an equation with grouping symbols.
- Solve real-world problems using equations with variables on both sides.

Solve an Equation with Variables on Both Sides

When a variable appears on both sides of the equation, we need to manipulate the equation so that all variable terms appear on one side, and only constants are left on the other.

Example 1

Dwayne was told by his chemistry teacher to measure the weight of an empty beaker using a balance. Dwayne found only one lb weights, and so devised the following way of balancing the scales.



Knowing that each weight is one lb, calculate the weight of one beaker.

Solution

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this fact. The unknown quantity, the weight of the beaker, will be our x . We can see that on the left hand scale we have one beaker and four weights. On the right scale, we have four beakers and three weights. The balancing of the scales is analogous to the balancing of the following equation:

$$x + 4 = 4x + 3$$

“One beaker plus 4 lbs **equals** 4 beakers plus 3 lbs”

To solve for the weight of the beaker, we want all the constants (numbers) on one side and all the variables

(terms with x in them) on the other side. Since there are more beakers on the right and more weights on the left, we'll try to move all the x terms (beakers) to the right, and the constants (weights) to the left.

First we subtract 3 from both sides to get $x + 1 = 4x$.

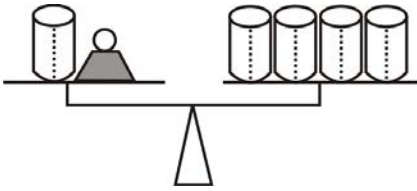
Then we subtract x from both sides to get $1 = 3x$.

Finally we divide by 3 to get $\frac{1}{3} = x$.

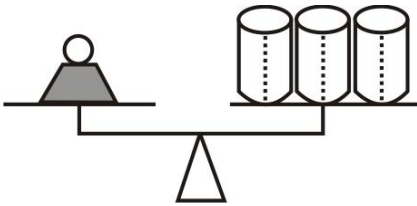
The weight of the beaker is one-third of a pound.

We can do the same with the real objects as we did with the equation. Just as we subtracted amounts from each side of the equation, we could remove a certain number of weights or beakers from each scale. Because we remove the same number of objects from each side, we know the scales will still balance.

First, we could remove three weights from each scale. This would leave one beaker and one weight on the left and four beakers on the right (in other words $x + 1 = 4x$):



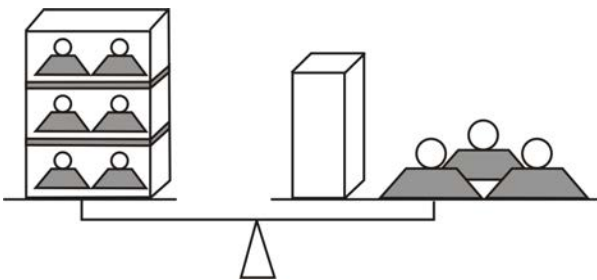
Then we could remove one beaker from each scale, leaving only one weight on the left and three beakers on the right, to get $1 = 3x$:



Looking at the balance, it is clear that the weight of one beaker is one-third of a pound.

Example 2

Sven was told to find the weight of an empty box with a balance. Sven found some one lb weights and five lb weights. He placed two one lb weights in three of the boxes and with a fourth empty box found the following way of balancing the scales:



Knowing that small weights are one lb and big weights are five lbs, calculate the weight of one box.

We know that the system balances, so the weights on each side must be equal. We can write an algebraic expression based on this equality. The unknown quantity—the weight of each empty box, in pounds—will be our x . A box with two 1 lb weights in it weighs $(x + 2)$ pounds. Our equation, based on the picture, is $3(x + 2) = x + 3(5)$.

Distributing the 3 and simplifying, we get $3x + 6 = x + 15$.

Subtracting x from both sides, we get $2x + 6 = 15$.

Subtracting 6 from both sides, we get $2x = 9$.

And finally we can divide by 2 to get $x = \frac{9}{2}$, or $x = 4.5$.

Each box weighs 4.5 lbs.

To see more examples of solving equations with variables on both sides of the equation, see the Khan Academy video at <http://www.youtube.com/watch?v=Zn-GbH2S0Dk>.

Solve an Equation with Grouping Symbols

As you've seen, we can solve equations with variables on both sides even when some of the variables are in parentheses; we just have to get rid of the parentheses, and then we can start combining like terms. We use the same technique when dealing with fractions: first we multiply to get rid of the fractions, and then we can shuffle the terms around by adding and subtracting.

Example 3

Solve $3x + 2 = \frac{5x}{3}$.

Solution

The first thing we'll do is get rid of the fraction. We can do this by multiplying both sides by 3, leaving $3(3x + 2) = 5x$.

Then we distribute to get rid of the parentheses, leaving $9x + 6 = 5x$.

We've already got all the constants on the left side, so we'll move the variables to the right side by subtracting $9x$ from both sides. That leaves us with $6 = -4x$.

And finally, we divide by -4 to get $-\frac{3}{2} = x$, or $x = -1.5$.

Example 4

Solve $7x + 2 = \frac{5x-3}{6}$.

Solution

Again we start by eliminating the fraction. Multiplying both sides by 6 gives us $6(7x + 2) = 5x - 3$, and distributing gives us $42x + 12 = 5x - 3$.

Subtracting $5x$ from both sides gives us $37x + 12 = -3$.

Subtracting 12 from both sides gives us $37x = -15$.

Finally, dividing by 37 gives us $x = -\frac{15}{37}$.

Example 5

Solve the following equation for x : $\frac{14x}{(x+3)} = 7$

Solution

The form of the left hand side of this equation is known as a **rational function** because it is the ratio of two other functions: $14x$ and $(x + 3)$. But we can solve it just like any other equation involving fractions.

First we multiply both sides by $(x + 3)$ to get rid of the fraction. Now our equation is $14x = 7(x + 3)$.

Then we distribute: $14x = 7x + 21$.

Then subtract $7x$ from both sides: $7x = 21$.

And divide by 7: $x = 3$.

Solve Real-World Problems Using Equations with Variables on Both Sides

Here's another chance to practice translating problems from words to equations. What is the equation asking? What is the **unknown** variable? What quantity will we use for our variable?

The text explains what's happening. Break it down into small, manageable chunks, and follow what's going on with our variable all the way through the problem.

More on Ohm's Law

Recall that the electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$ where R is the resistance measured in Ohms (Ω).

The resistance R of a number of components wired in a **series** (one after the other) is simply the sum of all the resistances of the individual components.

Example 6

In an attempt to find the resistance of a new component, a scientist tests it in series with standard resistors. A fixed voltage causes a 4.8 amp current in a circuit made up from the new component plus a 15Ω resistor in series. When the component is placed in a series circuit with a 50Ω resistor, the same voltage causes a 2.0 amp current to flow. Calculate the resistance of the new component.

This is a complex problem to translate, but once we convert the information into equations it's relatively straightforward to solve. First, we are trying to find the resistance of the new component (in Ohms, Ω). This is our x . We don't know the voltage that is being used, but we can leave that as a variable, V . Our first situation has a total resistance that equals the unknown resistance plus 15Ω . The current is 4.8 amps. Substituting into the formula $V = I \cdot R$, we get $V = 4.8(x + 15)$.

Our second situation has a total resistance that equals the unknown resistance plus 50Ω . The current is 2.0 amps. Substituting into the same equation, this time we get $V = 2(x + 50)$.

We know the voltage is fixed, so the V in the first equation must equal the V in the second. That means we can set the right-hand sides of the two equations equal to each other: $4.8(x + 15) = 2(x + 50)$. Then we can solve for x .

Distribute the constants first: $4.8x + 72 = 2x + 100$.

Subtract $2x$ from both sides: $2.8x + 72 = 100$.

Subtract 72 from both sides: $2.8x = 28$.

Divide by 2.8: $x = 10$.

The resistance of the component is 10Ω .

Lesson Summary

If an unknown variable appears on both sides of an equation, distribute as necessary. Then simplify the equation to have the unknown on only one side.

Review Questions

1. Solve the following equations for the unknown variable.

(a) $3(x - 1) = 2(x + 3)$

- (b) $7(x + 20) = x + 5$
- (c) $9(x - 2) = 3x + 3$
- (d) $2\left(a - \frac{1}{3}\right) = \frac{2}{5}\left(a + \frac{2}{3}\right)$
- (e) $\frac{2}{7}\left(t + \frac{2}{3}\right) = \frac{1}{5}\left(t - \frac{2}{3}\right)$
- (f) $\frac{1}{7}\left(v + \frac{1}{4}\right) = 2\left(\frac{3v}{2} - \frac{5}{2}\right)$
- (g) $\frac{v-4}{11} = \frac{2}{5} \cdot \frac{2v+1}{3}$
- (h) $\frac{z}{16} = \frac{2(3z+1)}{9}$
- (i) $\frac{q}{16} + \frac{q}{6} = \frac{(3q+1)}{9} + \frac{3}{2}$
- (j) $\frac{3}{x} = \frac{2}{x+1}$
- (k) $\frac{5}{2+p} = \frac{3}{p-8}$

2. Manoj and Tamar are arguing about a number trick they heard. Tamar tells Andrew to think of a number, multiply it by five and subtract three from the result. Then Manoj tells Andrew to think of a number, add five and multiply the result by three. Andrew says that whichever way he does the trick he gets the same answer.
 - (a) What was the number Andrew started with?
 - (b) What was the result Andrew got both times?
 - (c) Name another set of steps that would have resulted in the same answer if Andrew started with the same number.
3. Manoj and Tamar try to come up with a harder trick. Manoj tells Andrew to think of a number, double it, add six, and then divide the result by two. Tamar tells Andrew to think of a number, add five, triple the result, subtract six, and then divide the result by three.
 - (a) Andrew tries the trick both ways and gets an answer of 10 each time. What number did he start out with?
 - (b) He tries again and gets 2 both times. What number did he start out with?
 - (c) Is there a number Andrew can start with that will *not* give him the same answer both ways?
 - (d) **Bonus:** Name another set of steps that would give Andrew the same answer every time as he would get from Manoj's and Tamar's steps.
4. I have enough money to buy five regular priced CDs and have \$6 left over. However, all CDs are on sale today, for \$4 less than usual. If I borrow \$2, I can afford nine of them.
 - (a) How much are CDs on sale for today?
 - (b) How much would I have to borrow to afford nine of them if they weren't on sale?
5. Five identical electronics components were connected in series. A fixed but unknown voltage placed across them caused a 2.3 amp current to flow. When two of the components were replaced with standard 10Ω resistors, the current dropped to 1.9 amps. What is the resistance of each component?
6. Solve the following resistance problems. Assume the same voltage is applied to all circuits.
 - (a) Three unknown resistors plus 20Ω give the same current as one unknown resistor plus 70Ω .
 - (b) One unknown resistor gives a current of 1.5 amps and a 15Ω resistor gives a current of 3.0 amps.
 - (c) Seven unknown resistors plus 18Ω gives twice the current of two unknown resistors plus 150Ω .
 - (d) Three unknown resistors plus 1.5Ω gives a current of 3.6 amps and seven unknown resistors plus seven 12Ω resistors gives a current of 0.2 amps.

3.5 Ratios and Proportions

Learning Objectives

- Write and understand a ratio.

- Write and solve a proportion.
- Solve proportions using cross products.
- Solve problems using scale drawings.

Introduction

Nadia is counting out money with her little brother. She gives her brother all the nickels and pennies. She keeps the quarters and dimes for herself. Nadia has four quarters and six dimes. Her brother has fifteen nickels and five pennies and is happy because he has more coins than his big sister. How would you explain to him that he is actually getting a bad deal?

Write a ratio

A **ratio** is a way to compare two numbers, measurements or quantities. When we write a ratio, we divide one number by another and express the answer as a fraction. There are two distinct ratios in the problem above. For example, the ratio of the **number** of Nadia's coins to her brother's is $\frac{4+6}{15+5}$, or $\frac{10}{20} = \frac{1}{2}$. (Ratios should always be simplified.) In other words, Nadia has half as many coins as her brother.

Another ratio we could look at is the **value** of the coins. The value of Nadia's coins is $(4 \times 25) + (6 \times 10) = 160$ cents. The value of her brother's coins is $(15 \times 5) + (5 \times 1) = 80$ cents. The ratio of the **value** of Nadia's coins to her brother's is $\frac{160}{80} = \frac{2}{1}$. So the value of Nadia's money is twice the value of her brother's.

Notice that even though the denominator is one, we still write it out and leave the ratio as a fraction instead of a whole number. A ratio with a denominator of one is called a **unit rate**.

Example 1

The price of a Harry Potter Book on Amazon.com is \$10.00. The same book is also available used for \$6.50. Find two ways to compare these prices.

Solution

We could compare the numbers by expressing the difference between them: $\$10.00 - \$6.50 = \$3.50$.

We can also use a ratio to compare them: $\frac{10.00}{6.50} = \frac{100}{65} = \frac{20}{13}$ (after multiplying by 10 to remove the decimals, and then simplifying).

So we can say that **the new book is \$3.50 more than the used book**, or we can say that **the new book costs $\frac{20}{13}$ times as much as the used book**.

Example 2

A tournament size shuffleboard table measures 30 inches wide by 14 feet long. Compare the length of the table to its width and express the answer as a ratio.

Solution

We could just write the ratio as $\frac{14 \text{ feet}}{30 \text{ inches}}$. But since we're comparing two lengths, it makes more sense to convert all the measurements to the same units. 14 feet is $14 \times 12 = 168$ inches, so our new ratio is $\frac{168}{30} = \frac{28}{5}$.

Example 3

*A family car is being tested for fuel efficiency. It drives non-stop for 100 miles and uses 3.2 gallons of gasoline. Write the ratio of distance traveled to fuel used as a **unit rate**.*

Solution

The ratio of distance to fuel is $\frac{100 \text{ miles}}{3.2 \text{ gallons}}$. But a unit rate has to have a denominator of one, so to make

this ratio a unit rate we need to divide both numerator and denominator by 3.2. $\frac{\frac{100}{3.2} \text{ miles}}{\frac{3.2}{3.2} \text{ gallons}} = \frac{31.25 \text{ miles}}{1 \text{ gallon}}$ or **31.25 miles per gallon.**

Write and Solve a Proportion

When two ratios are equal to each other, we call it a proportion. For example, the equation $\frac{10}{5} = \frac{6}{9}$ is a proportion. We know it's true because we can reduce both fractions to $\frac{2}{3}$.

(Check this yourself to make sure!)

We often use proportions in science and business—for example, when scaling up the size of something. We generally use them to solve for an unknown, so we use algebra and label the unknown variable x .

Example 4

A small fast food chain operates 60 stores and makes \$1.2 million profit every year. How much profit would the chain make if it operated 250 stores?

Solution

First, we need to write a **ratio**: the ratio of profit to number of stores. That would be $\frac{\$1,200,000}{60}$.

Now we want to know how much profit 250 stores would make. If we label that profit x , then the ratio of profit to stores in that case is $\frac{x}{250}$.

Since we're assuming the profit is proportional to the number of stores, the ratios are equal and our proportion is $\frac{1,200,000}{60} = \frac{x}{250}$.

(Note that we can drop the units – not because they are the same in the numerator and denominator, but because they are the same on both sides of the equation.)

To solve this equation, first we simplify the left-hand fraction to get $20,000 = \frac{x}{250}$. Then we multiply both sides by 250 to get $5,000,000 = x$.

If the chain operated 250 stores, the annual profit would be 5 million dollars.

Example 5

A chemical company makes up batches of copper sulfate solution by adding 250 kg of copper sulfate powder to 1000 liters of water. A laboratory chemist wants to make a solution of identical concentration, but only needs 350 mL (0.35 liters) of solution. How much copper sulfate powder should the chemist add to the water?

Solution

The ratio of powder to water in the first case, in kilograms per liter, is $\frac{250}{1000}$, which reduces to $\frac{1}{4}$. In the second case, the unknown amount is how much powder to add. If we label that amount x , the ratio is $\frac{x}{0.35}$. So our proportion is $\frac{1}{4} = \frac{x}{0.35}$.

To solve for x , first we multiply both sides by 0.35 to get $\frac{0.35}{4} = x$, or $x = 0.0875$.

The mass of copper sulfate that the chemist should add is 0.0875 kg, or 87.5 grams.

Solve Proportions Using Cross Products

One neat way to simplify proportions is to cross multiply. Consider the following proportion:

$$\frac{16}{4} = \frac{20}{5}$$

If we want to eliminate the fractions, we could multiply both sides by 4 and then multiply both sides by 5. But suppose we just do both at once?

$$4 \times 5 \times \frac{16}{4} = 4 \times 5 \times \frac{20}{5}$$
$$5 \times 16 = 4 \times 20$$

Now comparing this to the proportion we started with, we see that the denominator from the left hand side ends up being multiplied by the numerator on the right hand side. You can also see that the denominator from the *right* hand side ends up multiplying the numerator on the *left* hand side.

In effect the two denominators have multiplied across the equal sign:

$$\frac{16}{4} = \frac{20}{5}$$

becomes $5 \times 16 = 4 \times 20$.

This movement of denominators is known as **cross multiplying**. It is extremely useful in solving proportions, especially when the unknown variable is in the denominator.

Example 6

Solve this proportion for x : $\frac{4}{3} = \frac{9}{x}$

Solution

Cross multiply to get $4x = 9 \times 3$, or $4x = 27$. Then divide both sides by 4 to get $x = \frac{27}{4}$, or $x = 6.75$.

Example 7

Solve the following proportion for x : $\frac{0.5}{3} = \frac{56}{x}$

Solution

Cross multiply to get $0.5x = 56 \times 3$, or $0.5x = 168$. Then divide both sides by 0.5 to get $x = 336$.

Solve Real-World Problems Using Proportions

Example 8

A cross-country train travels at a steady speed. It covers 15 miles in 20 minutes. How far will it travel in 7 hours assuming it continues at the same speed?

Solution

We've done speed problems before; remember that speed is just the ratio $\frac{\text{distance}}{\text{time}}$, so that ratio is the one we'll use for our proportion. We can see that the speed is $\frac{15 \text{ miles}}{20 \text{ minutes}}$, and that speed is also equal to $\frac{x \text{ miles}}{7 \text{ hours}}$.

To set up a proportion, we first have to get the units the same. 20 minutes is $\frac{1}{3}$ of an hour, so our proportion will be $\frac{15}{\frac{1}{3}} = \frac{x}{7}$. This is a very awkward looking ratio, but since we'll be cross multiplying, we can leave it as it is.

Cross multiplying gives us $7 \times 15 = \frac{1}{3}x$. Multiplying both sides by 3 then gives us $3 \times 7 \times 15 = x$, or $x = 315$.

The train will travel 315 miles in 7 hours.

Example 9

In the United Kingdom, Alzheimer's disease is said to affect one in fifty people over 65 years of age. If approximately 250000 people over 65 are affected in the UK, how many people over 65 are there in total?

Solution

The fixed ratio in this case is the 1 person in 50. The unknown quantity x is the total number of people over 65. Note that in this case we don't need to include the units, as they will cancel between the numerator and denominator.

Our proportion is $\frac{1}{50} = \frac{250000}{x}$. Each ratio represents $\frac{\text{people with Alzheimer's}}{\text{total people}}$.

Cross multiplying, we get $1 \cdot x = 250000 \cdot 50$, or $x = 12,500,000$.

There are approximately 12.5 million people over the age of 65 in the UK.

For some more advanced ratio problems and applications, watch the Khan Academy video at <http://www.youtube.com/watch?v=PASSD20cU0c>.

Scale and Indirect Measurement

One place where ratios are often used is in making maps. The **scale** of a map describes the relationship between distances on a map and the corresponding distances on the earth's surface. These measurements are expressed as a fraction or a ratio.

So far we have only written ratios as fractions, but outside of mathematics books, ratios are often written as two numbers separated by a colon (:). For example, instead of $\frac{2}{3}$, we would write 2:3.

Ratios written this way are used to express the relationship between a map and the area it represents. For example, a map with a scale of 1:1000 would be a map where one unit of measurement (such as a centimeter) on the map would represent 1000 of the same unit (1000 centimeters, or 10 meters) in real life.

Example 10

Anne is visiting a friend in London, and is using the map below to navigate from Fleet Street to Borough Road. She is using a 1:100,000 scale map, where 1 cm on the map represents 1 km in real life. Using a ruler, she measures the distance on the map as 8.8 cm. How far is the real distance from the start of her journey to the end?



Solution

The scale is the ratio of distance on the map to the corresponding distance in real life. Written as a fraction, it is $\frac{1}{100000}$. We can also write an equivalent ratio for the distance Anne measures on the map and the distance in real life that she is trying to find: $\frac{8.8}{x}$. Setting these two ratios equal gives us our proportion: $\frac{1}{100000} = \frac{8.8}{x}$. Then we can cross multiply to get $x = 880000$.

That's how many *centimeters* it is from Fleet Street to Borough Road; now we need to convert to kilometers. There are 100000 cm in a km, so we have to divide our answer by 100000.

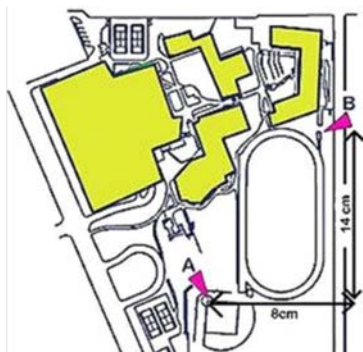
$$\frac{880000}{100000} = 8.8.$$

The distance from Fleet Street to Borough Road is 8.8 km.

In this problem, we could have just used our intuition: the $1 \text{ cm} = 1 \text{ km}$ scale tells us that any number of cm on the map is equal to the same number of km in real life. But not all maps have a scale this simple. You'll usually need to refer to the map scale to convert between measurements on the map and distances in real life!

Example 11

Antonio is drawing a map of his school for a project in math. He has drawn out the following map of the school buildings and the surrounding area



He is trying to determine the scale of his figure. He knows that the distance from the point marked A on the baseball diamond to the point marked B on the athletics track is 183 meters. Use the dimensions marked on the drawing to determine the scale of his map.

Solution

We know that the real-life distance is 183 m, and the scale is the ratio $\frac{\text{distance on map}}{\text{distance in real life}}$.

To find the distance on the map, we use Pythagoras' Theorem: $a^2 + b^2 = c^2$, where a and b are the horizontal and vertical lengths and c is the diagonal between points A and B .

$$\begin{aligned} 8^2 + 14^2 &= c^2 \\ 64 + 196 &= c^2 \\ 260 &= c^2 \\ \sqrt{260} &= c \\ 16.12 &\approx c \end{aligned}$$

So the distance on the map is about 16.12 cm. The distance in real life is 183 m, which is 18300 cm. Now we can divide:

$$\text{Scale} = \frac{16.12}{18300} \approx \frac{1}{1135.23}$$

The scale of Antonio's map is approximately 1:1100.

Another visual use of ratio and proportion is in **scale drawings**. Scale drawings (often called **plans**) are used extensively by architects. The equations governing scale are the same as for maps; the scale of a drawing is the ratio $\frac{\text{distance on diagram}}{\text{distance in real life}}$.

Example 12

Oscar is trying to make a scale drawing of the Titanic, which he knows was 883 ft long. He would like his drawing to be at a 1:500 scale. How many inches long does his sheet of paper need to be?

Solution

We can reason intuitively that since the scale is 1:500, the paper must be $\frac{883}{500} = 1.766$ feet long. Converting to inches means the length is $12(1.766) = 21.192$ inches.

Oscar's paper should be at least 22 inches long.

Example 13

The Rose Bowl stadium in Pasadena, California measures 880 feet from north to south and 695 feet from east to west. A scale diagram of the stadium is to be made. If 1 inch represents 100 feet, what would be the dimensions of the stadium drawn on a sheet of paper? Will it fit on a standard 8.5 × 11 inch sheet of paper?

Solution

Instead of using a proportion, we can simply use the following equation: (distance on diagram) = (distance in real life) × (scale). (We can derive this from the fact that scale = $\frac{\text{distance on diagram}}{\text{distance in real life}}$.)

Plugging in, we get

$$\text{height on paper} = 880 \text{ feet} \times \frac{1 \text{ inch}}{100 \text{ feet}} = 8.8 \text{ inches}$$

$$\text{width on paper} = 695 \text{ feet} \times \frac{1 \text{ inch}}{100 \text{ feet}} = 6.95 \text{ inches}$$

The scale diagram will be 8.8 in × 6.95 in. It will fit on a standard sheet of paper.

Lesson Summary

- A **ratio** is a way to compare two numbers, measurements or quantities by dividing one number by the other and expressing the answer as a fraction.
- A **proportion** is formed when two ratios are set equal to each other.
- **Cross multiplication** is useful for solving equations in the form of proportions. To cross multiply, multiply the bottom of each ratio by the top of the other ratio and set them equal. For instance, cross multiplying

$$\frac{11}{5} \times \frac{x}{3} \text{ results in } 11 \times 3 = 5x.$$

- **Scale** is a proportion that relates map distance to real life distance.

Review Questions

1. Write the following comparisons as ratios. Simplify fractions where possible.
 - (a) \$150 to \$3
 - (b) 150 boys to 175 girls
 - (c) 200 minutes to 1 hour
 - (d) 10 days to 2 weeks
2. Write the following ratios as a unit rate.
 - (a) 54 hotdogs to 12 minutes
 - (b) 5000 lbs to 250 square inches
 - (c) 20 computers to 80 students
 - (d) 180 students to 6 teachers
 - (e) 12 meters to 4 floors
 - (f) 18 minutes to 15 appointments
3. Solve the following proportions.
 - (a) $\frac{13}{6} = \frac{5}{x}$

- (b) $\frac{1.25}{6} = \frac{3.6}{x}$
 (c) $\frac{6}{19} = \frac{x}{11}$
 (d) $\frac{1}{x} = \frac{0.01}{5}$
 (e) $\frac{300}{4} = \frac{x}{99}$
 (f) $\frac{2.75}{9} = \frac{x}{(\frac{2}{9})}$
 (g) $\frac{1.3}{4} = \frac{x}{1.3}$
 (h) $\frac{0.1}{1.01} = \frac{1.9}{x}$

4. A restaurant serves 100 people per day and takes in \$908. If the restaurant were to serve 250 people per day, how much money would it take in?
5. The highest mountain in Canada is Mount Yukon. It is $\frac{298}{67}$ the size of Ben Nevis, the highest peak in Scotland. Mount Elbert in Colorado is the highest peak in the Rocky Mountains. Mount Elbert is $\frac{220}{67}$ the height of Ben Nevis and $\frac{11}{12}$ the size of Mont Blanc in France. Mont Blanc is 4800 meters high. How high is Mount Yukon?
6. At a large high school it is estimated that two out of every three students have a cell phone, and one in five of all students have a cell phone that is one year old or less. Out of the students who own a cell phone, what proportion owns a phone that is more than one year old?
7. Use the map in Example 10. Using the scale printed on the map, determine the distances (rounded to the nearest half km) between:
- (a) Points 1 and 4
 (b) Points 22 and 25
 (c) Points 18 and 13
 (d) Tower Bridge and London Bridge

3.6 Percent Problems

Learning Objectives

- Find a percent of a number.
- Use the percent equation.
- Find the percent of change.

Introduction

A **percent** is simply a ratio with a base unit of 100. When we write a ratio as a fraction, the percentage we want to represent is the numerator, and the denominator is 100. For example, 43% is another way of writing $\frac{43}{100}$. $\frac{43}{1000}$, on the other hand, is equal to $\frac{4.3}{100}$, so it would be equivalent to 4.3%. $\frac{2}{5}$ is equal to $\frac{40}{100}$, or 40%. To convert any fraction to a percent, just convert it to an equivalent fraction with a denominator of 100, and then take the numerator as your percent value.

To convert a percent to a decimal, just move the decimal point two spaces to the right:

$$\begin{aligned} 67\% &= 0.67 \\ 0.2\% &= 0.002 \\ 150\% &= 1.5 \end{aligned}$$

And to convert a decimal to a percent, just move the decimal point two spaces to the left:

$$\begin{aligned}2.3 &= 230\% \\0.97 &= 97\% \\0.00002 &= 0.002\%\end{aligned}$$

Finding and Converting Percentages

Before we work with percentages, we need to know how to convert between percentages, decimals and fractions.

Converting percentages to fractions is the easiest. The word “percent” simply means “per 100”—so, for example, 55% means 55 per 100, or $\frac{55}{100}$. This fraction can then be simplified to $\frac{11}{20}$.

Example 1

Convert 32.5% to a fraction.

Solution

32.5% is equal to 32.5 per 100, or $\frac{32.5}{100}$. To reduce this fraction, we first need to multiply it by $\frac{10}{10}$ to get rid of the decimal point. $\frac{325}{1000}$ then reduces to $\frac{13}{40}$.

Converting fractions to percentages can be a little harder. To convert a fraction directly to a percentage, you need to express it as an equivalent fraction with a denominator of 100.

Example 2

Convert $\frac{7}{8}$ to a percent.

Solution

To get the denominator of this fraction equal to 100, we have to multiply it by 12.5. Multiplying the numerator by 12.5 also, we get $\frac{87.5}{100}$, which is equivalent to 87.5%.

But what about a fraction like $\frac{1}{6}$, where there’s no convenient number to multiply the denominator by to get 100? In a case like this, it’s easier to do the division problem suggested by the fraction in order to convert the fraction to a decimal, and *then* convert the decimal to a percent. 1 divided by 6 works out to 0.166666.... Moving the decimal two spaces to the right tells us that this is equivalent to about 16.7%.

Why can we convert from decimals to percents just by moving the decimal point? Because of what decimal places represent. 0.1 is another way of representing one tenth, and 0.01 is equal to one hundredth—and one hundredth is one percent. By the same logic, 0.02 is 2 percent, 0.35 is 35 percent, and so on.

Example 3

Convert 2.64 to a percent.

Solution

To convert to a percent, simply move the decimal two places to the right. $2.64 = 264\%$.

Does a percentage greater than 100 even make sense? Sure it does—percentages greater than 100 come up in real life all the time. For example, a business that made 10 million dollars last year and 13 million dollars this year would have made 130% as much money this year as it did last year.

The only situation where a percentage greater than 100 doesn’t make sense is when you’re talking about dividing up something that you only have a fixed amount of—for example, if you took a survey and found that 56% of the respondents gave one answer and 72% gave another answer (for a total of 128%), you’d know something went wrong with your math somewhere, because there’s no way you could have gotten answers from more than 100% of the people you surveyed.

Converting percentages to decimals is just as easy as converting decimals to percentages—simply move the decimal to the left instead of to the right.

Example 4

Convert 58% to a decimal.

Solution

The decimal point here is invisible—it's right after the 8. So moving it to the left two places gives us 0.58.

It can be hard to remember which way to move the decimal point when converting from decimals to percents or vice versa. One way to check if you're moving it the right way is to check whether your answer is a bigger or smaller number than you started out with. If you're converting from percents to decimals, you should end up with a smaller number—just think of how a number like 50 percent, where 50 is **greater** than 1, represents a fraction like $\frac{1}{2}$ (or 0.50 in decimal form), where $\frac{1}{2}$ is **less** than 1. Conversely, if you're converting from decimals to percents, you should end up with a bigger number.

One way you might remember this is by remembering that a percent sign is bigger than a decimal point—so percents should be bigger numbers than decimals.

Example 5

Convert 3.4 to a percent.

Solution

If you move the decimal point to the left, you get 0.034%. That's a smaller number than you started out with, but you're moving from decimals to percents, so you want the number to get bigger, not smaller. Move it to the right instead to get 340%.

Now let's try another fraction.

Example 6

Convert $\frac{2}{7}$ to a percent.

Solution

$\frac{2}{7}$ doesn't convert easily unless you change it to a decimal first. 2 divided by 7 is approximately 0.285714..., and moving the decimal and rounding gives us 28.6%.

The following Khan Academy video shows several more examples of finding percents and might be useful for further practice: http://www.youtube.com/watch?v=_SpE4hQ8D_o.

Use the Percent Equation

The percent equation is often used to solve problems. It goes like this:

$$\text{Rate} \times \text{Total} = \text{Part}$$

or

$$R\% \text{ of Total is Part}$$

Rate is the ratio that the percent represents ($R\%$ in the second version).

Total is often called the *base unit*.

Part is the amount we are comparing with the base unit.

Example 7

Find 25% of \$80.

Solution

We are looking for the **part**. The **total** is \$80. ‘of’ means multiply. $R\%$ is 25%, so we can use the second form of the equation: 25% of \$80 is Part, or $0.25 \times 80 = \text{Part}$.

$0.25 \times 80 = 20$, so the Part we are looking for is **\$20**.

Example 8

Express \$90 as a percentage of \$160.

Solution

This time we are looking for the **rate**. We are given the **part** (\$90) and the **total** (\$160). Using the rate equation, we get $\text{Rate} \times 160 = 90$. Dividing both sides by 160 tells us that the rate is 0.5625, or 56.25%.

Example 9

\$50 is 15% of what total sum?

This time we are looking for the **total**. We are given the **part** (\$50) and the **rate** (15%, or 0.15). Using the rate equation, we get $0.15 \times \text{Total} = 50$. Dividing both sides by 0.15, we get $\text{Total} = \frac{50}{0.15} \approx 333.33$. So **\$50 is 15% of \$333.33**.

Find Percent of Change

A useful way to express changes in quantities is through percents. You’ve probably seen signs such as “20% extra free,” or “save 35% today.” When we use percents to represent a change, we generally use the formula

$$\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

or

$$\frac{\text{percent change}}{100} = \frac{\text{actual change}}{\text{original amount}}$$

This means that a **positive** percent change is an **increase**, while a **negative** change is a **decrease**.

Example 10

A school of 500 students is expecting a 20% increase in students next year. How many students will the school have?

Solution

First let’s solve this using the first formula. Since the 20% change is an increase, we represent it in the formula as 20 (if it were a decrease, it would be -20.) Plugging in all the numbers, we get

$$20\% = \frac{\text{final amount} - 500}{500} \times 100\%$$

Dividing both sides by 100%, we get $0.2 = \frac{\text{final amount} - 500}{500}$.

Multiplying both sides by 500 gives us $100 = \text{final amount} - 500$.

Then adding 500 to both sides gives us 600 as the final number of students.

How about if we use the second formula? Then we get $\frac{20}{100} = \frac{\text{actual change}}{500}$. (Reducing the first fraction to $\frac{1}{5}$ will make the problem easier, so let’s rewrite the equation as $\frac{1}{5} = \frac{\text{actual change}}{500}$)

Cross multiplying is our next step; that gives us $500 = 5 \times (\text{actual change})$. Dividing by 5 tells us the change is equal to 100. We were told this was an increase, so if we start out with 500 students, after an increase of 100 we know there will be a total of 600.

Markup

A **markup** is an increase from the price a store pays for an item from its supplier to the retail price it charges to the public. For example, a 100% mark-up (commonly known in business as *keystone*) means that the price is doubled. Half of the retail price covers the cost of the item from the supplier, half is profit.

Example 11

A furniture store places a 30% markup on everything it sells. It offers its employees a 20% discount from the sales price. The employees are demanding a 25% discount, saying that the store would still make a profit. The manager says that at a 25% discount from the sales price would cause the store to lose money. Who is right?

Solution

We'll consider this problem two ways. First, let's consider an item that the store buys from its supplier for a certain price, say \$1000. The markup would be 30% of 1000, or \$300, so the item would sell for \$1300 and the store would make a \$300 profit.

And what if an employee buys the product? With a discount of 20%, the employee would pay 80% of the \$1300 retail price, or $0.8 \times \$1300 = \1040 .

But with a 25% discount, the employee would pay 75% of the retail price, or $0.75 \times \$1300 = \975 .

So with a 20% employee discount, the store still makes a \$40 profit on the item they bought for \$1000—but with a 25% employee discount, the store loses \$25 on the item.

Now let's use algebra to see how this works for an item of any price. If x is the price of an item, then the store's markup is 30% of x , or $0.3x$, and the retail price of the item is $x + 0.3x$, or $1.3x$. An employee buying the item at a 20% discount would pay $0.8 \times 1.3x = 1.04x$, while an employee buying it at a 25% discount would pay $0.75 \times 1.3x = 0.975x$.

So the manager is right: a 20% employee discount still allows the store to make a profit, while a 25% employee discount would cause the store to lose money.

It may not seem to make sense that the store would lose money after applying a 30% markup and only a 25% discount. The reason it does work out that way is that the discount is bigger in absolute dollars after the markup is factored in. That is, an employee getting 25% off an item is getting 25% off the original price *plus* 25% off the 30% markup, and those two numbers together add up to more than 30% of the original price.

Solve Real-World Problems Using Percents

Example 12

*In 2004 the US Department of Agriculture had 112071 employees, of which 87846 were Caucasian. Of the remaining minorities, African-American and Hispanic employees had the two largest demographic groups, with 11754 and 6899 employees respectively.**

- Calculate the total percentage of minority (non-Caucasian) employees at the USDA.*
- Calculate the percentage of African-American employees at the USDA.*
- Calculate the percentage of minority employees who were neither African-American nor Hispanic.*

Solution

a) Use the percent equation $\text{Rate} \times \text{Total} = \text{Part}$.

The *total* number of employees is 112071. We know that the number of Caucasian employees is 87846, which means that there must be $112071 - 87846 = 24225$ non-Caucasian employees. This is the *part*. Plugging in the total and the part, we get $\text{Rate} \times 112071 = 24225$.

Divide both sides by 112071 to get $\text{Rate} = \frac{24225}{112071} \approx 0.216$. Multiply by 100 to get this as a percent: 21.6%.

21.6% of USDA employees in 2004 were from minority groups.

b) Here, the total is still 112071 and the part is 11754, so we have $\text{Rate} \times 112071 = 11754$. Dividing, we get $\text{Rate} = \frac{11754}{112071} \approx 0.105$, or 10.5%.

10.5% of USDA employees in 2004 were African-American.

c) Here, our total is just the number of non-Caucasian employees, which we found out is 24225. Subtracting the African-American and Hispanic employees leaves $24225 - 11754 - 6899 = 5572$ employees in the group we're looking at.

So with 24225 for the whole and 5572 for the part, our equation is $\text{Rate} \times 24225 = 5572$, or $\text{Rate} = \frac{5572}{24225} \approx 0.230$, or 23%.

23% of USDA minority employees in 2004 were neither African-American nor Hispanic.

Example 13

In 1995 New York had 18136000 residents. There were 827025 reported crimes, of which 152683 were violent. By 2005 the population was 19254630 and there were 85839 violent crimes out of a total of 491829 reported crimes. (Source: New York Law Enforcement Agency Uniform Crime Reports.) Calculate the percentage change from 1995 to 2005 in:

a) *Population of New York*

b) *Total reported crimes*

c) *violent crimes*

Solution

This is a percentage change problem. Remember the formula for percentage change:

$$\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$$

In these problems, the final amount is the 2005 statistic, and the initial amount is the 1995 statistic.

a) Population:

$$\begin{aligned} \text{Percent change} &= \frac{19254630 - 18136000}{18136000} \times 100\% \\ &= \frac{1118630}{18136000} \times 100\% \\ &\approx 0.0617 \times 100\% \\ &= 6.17\% \end{aligned}$$

The population grew by 6.17%.

b) Total reported crimes:

$$\begin{aligned}
 \text{Percent change} &= \frac{491829 - 827025}{827025} \times 100\% \\
 &= \frac{-335196}{827025} \times 100\% \\
 &\approx -0.4053 \times 100\% \\
 &= -40.53\%
 \end{aligned}$$

The total number of reported crimes fell by **40.53%**.

c) Violent crimes:

$$\begin{aligned}
 \text{Percent change} &= \frac{85839 - 152683}{152683} \times 100\% \\
 &= \frac{-66844}{152683} \times 100\% \\
 &\approx -0.4377 \times 100\% \\
 &= -43.77\%
 \end{aligned}$$

The total number of violent crimes fell by 43.77%.

Lesson Summary

- A **percent** is simply a ratio with a base unit of 100—for example, $13\% = \frac{13}{100}$.
- The **percent equation** is $\text{Rate} \times \text{Total} = \text{Part}$, or $R\%$ of Total is Part.
- The percent change equation is $\text{Percent change} = \frac{\text{final amount} - \text{original amount}}{\text{original amount}} \times 100\%$. A **positive** percent change means the value **increased**, while a **negative** percent change means the value **decreased**.

Review Questions

- Express the following decimals as a percent.
 - 0.011
 - 0.001
 - 0.91
 - 1.75
 - 20
- Express the following percentages in decimal form.
 - 15%
 - 0.08%
 - 222%
 - 3.5%
 - 341.9%
- Express the following fractions as a percent (round to two decimal places when necessary).
 - $\frac{1}{6}$
 - $\frac{5}{24}$
 - $\frac{6}{7}$

- (d) $\frac{11}{7}$
(e) $\frac{13}{97}$
4. Express the following percentages as a reduced fraction.
- (a) 11%
(b) 65%
(c) 16%
(d) 12.5%
(e) 87.5%
5. Find the following.
- (a) 30% of 90
(b) 16.7% of 199
(c) 11.5% of 10.01
(d) $y\%$ of $3x$
6. A TV is advertised on sale. It is 35% off and now costs \$195. What was the pre-sale price?
7. An employee at a store is currently paid \$9.50 per hour. If she works a full year she gets a 12% pay raise. What will her new hourly rate be after the raise?
8. Store *A* and Store *B* both sell bikes, and both buy bikes from the same supplier at the same prices. Store *A* has a 40% mark-up for their prices, while store *B* has a 250% mark-up. Store *B* has a permanent sale and will always sell at 60% off the marked-up prices. Which store offers the better deal?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9613>.

Chapter 4

Graphs of Equations and Functions

4.1 The Coordinate Plane

Learning Objectives

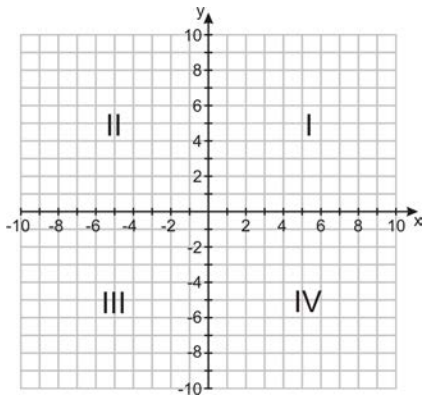
- Identify coordinates of points.
- Plot points in a coordinate plane.
- Graph a function given a table.
- Graph a function given a rule.

Introduction

Lydia lives 2 blocks north and one block east of school; Travis lives three blocks south and two blocks west of school. What's the shortest line connecting their houses?

The Coordinate Plane

We've seen how to represent numbers using number lines; now we'll see how to represent sets of numbers using a **coordinate plane**. The coordinate plane can be thought of as two number lines that meet at right angles. The horizontal line is called the **x -axis** and the vertical line is the **y -axis**. Together the lines are called the **axes**, and the point at which they cross is called the **origin**. The axes split the coordinate plane into four **quadrants**, which are numbered sequentially (I, II, III, IV) moving counter-clockwise from the upper right.



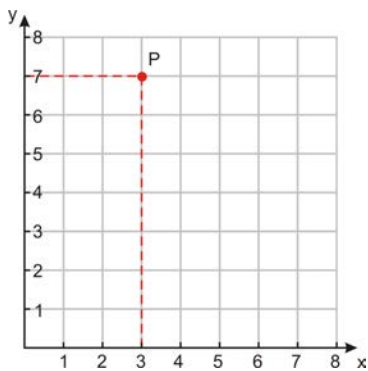
Identify Coordinates of Points

When given a point on a coordinate plane, it's easy to determine its **coordinates**. The coordinates of a point are two numbers - written together they are called an **ordered pair**. The numbers describe how far along the x -axis and y -axis the point is. The ordered pair is written in parentheses, with the x -**coordinate** (also called the **abscissa**) first and the y -**coordinate** (or the **ordinate**) second.

$(1, 7)$	An ordered pair with an x - value of one and a y - value of seven
$(0, 5)$	An ordered pair with an x - value of zero and a y - value of five
$(-2.5, 4)$	An ordered pair with an x - value of -2.5 and a y - value of four
$(-107.2, -.005)$	An ordered pair with an x - value of -107.2 and a y - value of $-.005$

Identifying coordinates is just like reading points on a number line, except that now the points do not actually lie **on** the number line! Look at the following example.

Example 1



Find the coordinates of the point labeled P in the diagram above

Solution

Imagine you are standing at the origin (the point where the x -axis meets the y -axis). In order to move to a position where P was directly above you, you would move 3 units to the **right** (we say this is in the **positive** x -direction).

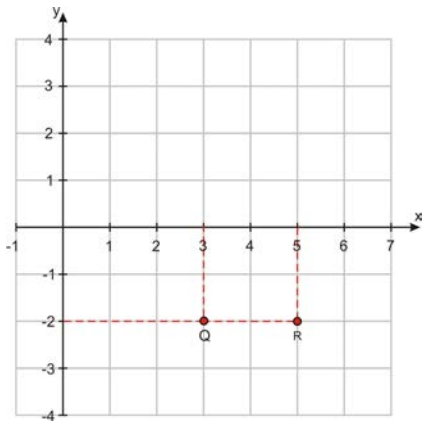
The x -coordinate of P is $+3$.

Now if you were standing at the 3 marker on the x -axis, point P would be 7 units **above** you (above the axis means it is in the **positive** y direction).

The y -coordinate of P is $+7$.

The coordinates of point P are $(3, 7)$.

Example 2



Find the coordinates of the points labeled Q and R in the diagram to the right.

Solution

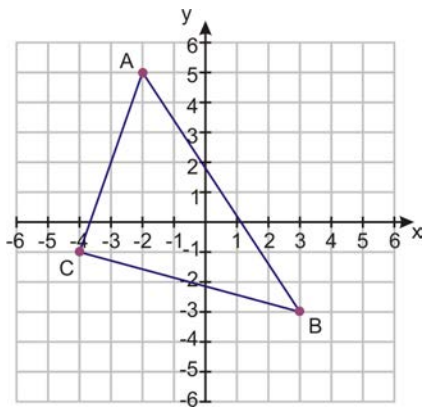
In order to get to Q we move three units to the right, in the positive x -direction, then two units **down**. This time we are moving in the **negative** y -direction. The x -coordinate of Q is $+3$, the y -coordinate of Q is -2 .

The coordinates of R are found in a similar way. The x -coordinate is $+5$ (five units in the positive x -direction) and the y -coordinate is again -2 .

The coordinates of Q are $(3, -2)$. The coordinates of R are $(5, -2)$.

Example 3

Triangle ABC is shown in the diagram to the right. Find the coordinates of the vertices A, B and C .



Point A :

x - coordinate = -2

y - coordinate = $+5$

Point B :

x - coordinate = $+3$

y - coordinate = -3

Point C:

x - coordinate = -4

y - coordinate = -1

Solution

$A(-2, 5)$

$B(3, -3)$

$C(-4, -1)$

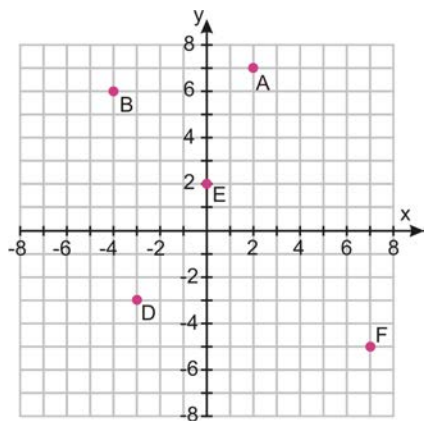
Plot Points in a Coordinate Plane

Plotting points is simple, once you understand how to read coordinates and read the scale on a graph. As a note on scale, in the next two examples pay close attention to the labels on the axes.

Example 4

Plot the following points on the coordinate plane.

$A(2, 7)$ $B(-4, 6)$ $D(-3, -3)$ $E(0, 2)$ $F(7, -5)$



Point $A(2, 7)$ is 2 units right, 7 units up. It is in Quadrant I.

Point $B(-4, 6)$ is 4 units left, 6 units up. It is in Quadrant II.

Point $D(-3, -3)$ is 3 units left, 3 units down. It is in Quadrant III.

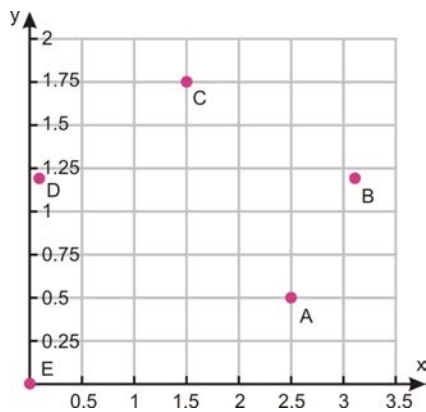
Point $E(0, 2)$ is 2 units up from the origin. It is right on the y -axis, between Quadrants I and II.

Point $F(7, -5)$ is 7 units right, 5 units down. It is in Quadrant IV.

Example 5

Plot the following points on the coordinate plane.

$A(2.5, 0.5)$ $B(\pi, 1.2)$ $C(2, 1.75)$ $D(0.1, 1.2)$ $E(0, 0)$



Here we see the importance of choosing the right scale and range for the graph. In Example 4, our points were scattered throughout the four quadrants. In this case, all the coordinates are positive, so we don't need to show the negative values of x or y . Also, there are no x -values bigger than about 3.14, and 1.75 is the largest value of y . We can therefore show just the part of the coordinate plane where $0 \leq x \leq 3.5$ and $0 \leq y \leq 2$.

Here are some other important things to notice about this graph:

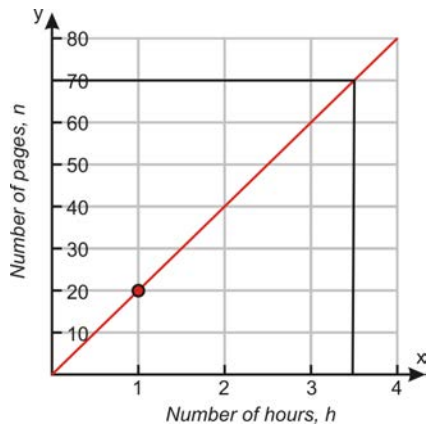
- The tick marks on the axes don't correspond to unit increments (i.e. the numbers do not go up by one each time). This is so that we can plot the points more precisely.
- The scale on the x -axis is different than the scale on the y -axis, so distances that look the same on both axes are actually greater in the x -direction. Stretching or shrinking the scale in one direction can be useful when the points we want to plot are farther apart in one direction than the other.

For more practice locating and naming points on the coordinate plane, try playing the Coordinate Plane Game at <http://www.math-play.com/Coordinate%20Plane%20Game/Coordinate%20Plane%20Game.html>.

Graph a Function Given a Table

Once we know how to plot points on a coordinate plane, we can think about how we'd go about plotting a relationship between x - and y -values. So far we've just been plotting sets of ordered pairs. A set like that is a **relation**, and there isn't necessarily a relationship between the x -values and y -values. If there is a relationship between the x - and y -values, and each x -value corresponds to exactly one y -value, then the relation is called a **function**. Remember that a function is a particular way to relate one quantity to another.

If you're reading a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either $n = 20h$ or $h = \frac{n}{20}$, where h is the number of hours you spend reading and n is the number of pages you read. To find out, for example, how many pages you could read in $3\frac{1}{2}$ hours, or how many hours it would take you to read 46 pages, you could use one of those formulas. Or, you could make a graph of the function:



Once you know how to graph a function like this, you can simply read the relationship between the x - and y -values off the graph. You can see in this case that you could read 70 pages in $3\frac{1}{2}$ hours, and it would take you about $2\frac{1}{3}$ hours to read 46 pages.

Generally, the graph of a **function** appears as a line or curve that goes through all points that have the relationship that the function describes. If the domain of the function (the set of x -values we can plug into the function) is all real numbers, then we call it a **continuous function**. If the domain of the function is a particular set of values (such as whole numbers only), then it is called a **discrete function**. The graph will be a series of dots, but they will still often fall along a line or curve.

In graphing equations, we assume the domain is all real numbers, unless otherwise stated. Often, though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. But sometimes we'll still draw the graph as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

Example 6

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

Table 4.1:

Number of Friends	Number of Presents
0	4
1	5
2	6
3	7
4	8
5	9

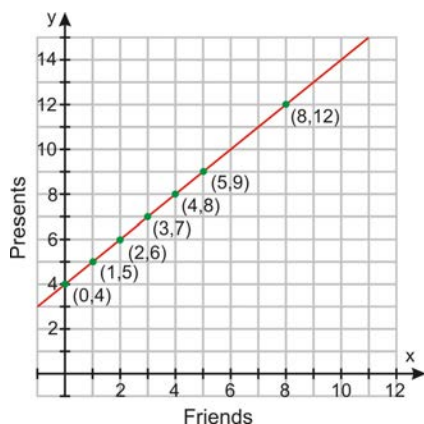
The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependent variable is. Clearly in this case, the number of friends can vary independently, but the number of presents must depend on the number of friends who show up.

So we'll plot friends on the x -axis and presents on the y -axis. Let's add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

Table 4.2:

Friends (x)	Presents (y)	Coordinates (x, y)
0	4	(0, 4)
1	5	(1, 5)
2	6	(2, 6)
3	7	(3, 7)
4	8	(4, 8)
5	9	(5, 9)

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we only need to show points in Quadrant I. Now we need to choose a suitable scale for the x - and y -axes. We only need to consider eight friends (look again at the question to confirm this), but it always pays to allow a little extra room on your graph. We also need the y -scale to accommodate the presents for eight people. We can see that this is still going to be under 20!



The scale of this graph has room for up to 12 friends and 15 presents. This will be fine, but there are many other scales that would be equally good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lie on a straight line, so the function that describes the relationship between x and y will be linear. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a discrete problem since Sarah can only invite a positive whole number of friends. For instance, it would be impossible for 2.4 or -3 friends to show up. So although the line helps us see where the other values of the function are, the only points on the line that actually *are* values of the function are the ones with positive whole-number coordinates.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah would get if eight friends come to her party. Don't forget that x represents the number of friends and y represents the number of presents. If we look at the graph where $x = 8$, we can see that the function has a y -value of 12.

Solution

If 8 friends show up, Sarah will receive a total of 12 presents.

Graph a Function Given a Rule

If we are given a rule instead of a table, we can proceed to graph the function in either of two ways. We will use the following example to show each way.

Example 7

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

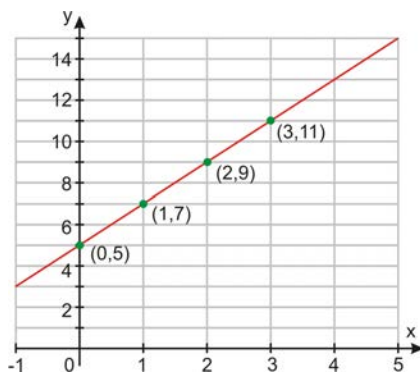
Method One - Construct a Table of Values

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few (x,y) pairs. We'll use 0, 1, 2 and 3 for x -values.

Table 4.3:

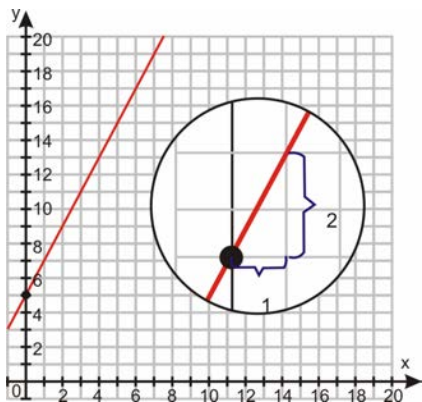
x	y
0	5
1	7
2	9
3	11

Next, we plot the points and join them with a line.



This method is nice and simple—especially with linear relationships, where we don't need to plot more than two or three points to see the shape of the graph. In this case, the function is continuous because the domain is all real numbers—that is, Ali could think of any real number, even though he may only be thinking of positive whole numbers.

Method Two - Intercept and Slope



Another way to graph this function (one that we'll learn in more detail in a later lesson) is the slope-intercept method. To use this method, follow these steps:

1. Find the y value when $y = 0$.

$y(0) = 2 \cdot 0 + 5 = 5$, so our y -intercept is $(0, 5)$.

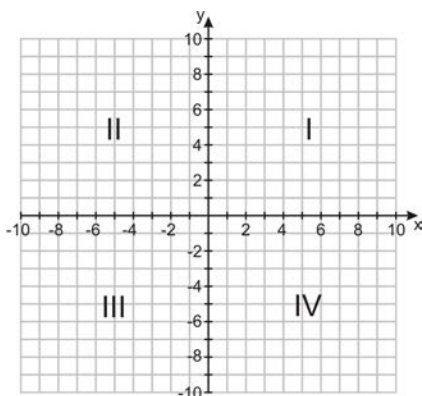
2. Look at the coefficient multiplying the x .

Every time we increase x by one, y increases by two, so our slope is $+2$.

3. Plot the line with the given slope that goes through the intercept. We start at the point $(0, 5)$ and move over one in the x -direction, then up two in the y -direction. This gives the slope for our line, which we extend in both directions.

We will properly examine this last method later in this chapter!

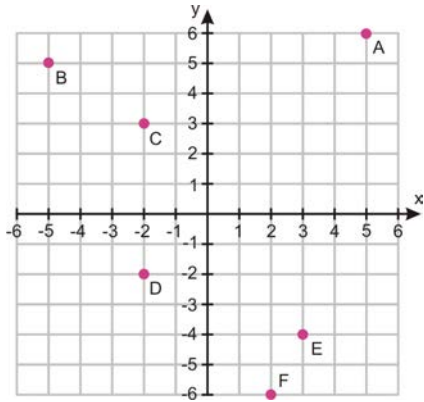
Lesson Summary



- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the x -axis) and a vertical number line (the y -axis). The **origin** is the point where these two lines meet. Four areas, or **quadrants**, are formed as shown in the diagram above.
- Each point on the coordinate plane has a set of **coordinates**, two numbers written as an **ordered pair** which describe how far along the x -axis and y -axis the point is. The x -**coordinate** is always written first, then the y -**coordinate**, in the form (x, y) .
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.

Review Questions

1. Identify the coordinates of each point, $A - F$, on the graph below.



2. Draw a line on the above graph connecting point B with the origin. Where does that line intersect the line connecting points C and D ?
3. Plot the following points on a graph and identify which quadrant each point lies in:
- (a) $(4, 2)$
 - (b) $(-3, 5.5)$
 - (c) $(4, -4)$
 - (d) $(-2, -3)$
4. Without graphing the following points, identify which quadrant each lies in:
- (a) $(5, 3)$
 - (b) $(-3, -5)$
 - (c) $(-4, 2)$
 - (d) $(2, -4)$
5. Consider the graph of the equation $y = 3$. Which quadrants does it pass through?
6. Consider the graph of the equation $y = x$. Which quadrants does it pass through?
7. Consider the graph of the equation $y = x + 3$. Which quadrants does it pass through?
8. The point $(4, 0)$ is on the boundary between which two quadrants?
9. The point $(0, -5)$ is on the boundary between which two quadrants?
10. If you moved the point $(3, 2)$ five units to the left, what quadrant would it be in?
11. The following three points are three vertices of square $ABCD$. Plot them on a graph, then determine what the coordinates of the fourth point, D , would be. Plot that point and label it.

$$A(-4, -4) \quad B(3, -4) \quad C(3, 3)$$

12. In what quadrant is the center of the square from problem 10? (You can find the center by drawing the square's diagonals.)
13. What point is halfway between $(1, 3)$ and $(1, 5)$?
14. What point is halfway between $(2, 8)$ and $(6, 8)$?
15. What point is halfway between the origin and $(10, 4)$?
16. What point is halfway between $(3, -2)$ and $(-3, 2)$?
17. Becky has a large bag of M&Ms that she knows she should share with Jaeyun. Jaeyun has a packet of Starburst. Becky tells Jaeyun that for every Starburst he gives her, she will give him three M&Ms in return. If x is the number of Starburst that Jaeyun gives Becky, and y is the number of M&Ms he gets in return, then complete each of the following.
- (a) Write an algebraic rule for y in terms of x .
 - (b) Make a table of values for y with x -values of 0, 1, 2, 3, 4, 5.

- (c) Plot the function linking x and y on the following scale: $0 \leq x \leq 10$, $0 \leq y \leq 10$.

4.2 Graphs of Linear Equations

Learning Objectives

- Graph a linear function using an equation.
- Write equations and graph horizontal and vertical lines.
- Analyze graphs of linear functions and read conversion graphs.

Introduction

You're stranded downtown late at night with only \$8 in your pocket, and your home is 6 miles away. Two cab companies serve this area; one charges \$1.20 per mile with an additional \$1 fee, and the other charges \$0.90 per mile with an additional \$2.50 fee. Which cab will be able to get you home?

Graph a Linear Equation

At the end of Lesson 4.1 we looked at ways to graph a function from a rule. A rule is a way of writing the relationship between the two quantities we are graphing. In mathematics, we tend to use the words **formula** and **equation** to describe the rules we get when we express relationships algebraically. Interpreting and graphing these equations is an important skill that you'll use frequently in math.

Example 1

A taxi costs more the further you travel. Taxis usually charge a fee on top of the per-mile charge to cover hire of the vehicle. In this case, the taxi charges \$3 as a set fee and \$0.80 per mile traveled. Here is the equation linking the cost in dollars (y) to hire a taxi and the distance traveled in miles (x).

$$y = 0.8x + 3$$

Graph the equation and use your graph to estimate the cost of a seven-mile taxi ride.

Solution

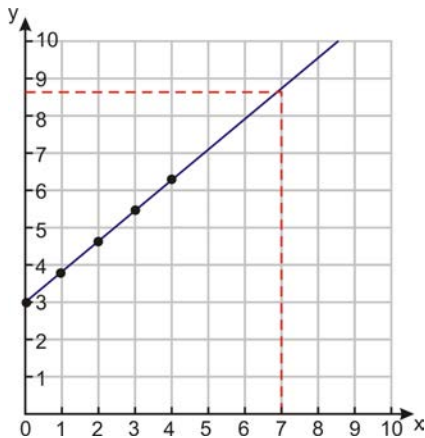
We'll start by making a table of values. We will take a few values for x (0, 1, 2, 3, and 4), find the corresponding y -values, and then plot them. Since the question asks us to find the cost for a seven-mile journey, we need to choose a scale that can accommodate this.

First, here's our table of values:

Table 4.4:

x	y
0	3
1	3.8
2	4.6
3	5.4
4	6.2

And here's our graph:



To find the cost of a seven-mile journey, first we find $x = 7$ on the horizontal axis and draw a line up to our graph. Next, we draw a horizontal line across to the y -axis and read where it hits. It appears to hit around half way between $y = 8$ and $y = 9$. Let's call it 8.5.

A seven mile taxi ride would cost approximately \$8.50 (\$8.60 exactly).

Here are some things you should notice about this graph and the formula that generated it:

- The graph is a straight line (this means that the equation is **linear**), although the function is **discrete** and really just consists of a series of points.
- The graph crosses the y -axis at $y = 3$ (notice that there's $a + 3$ in the equation—that's not a coincidence!). This is the base cost of the taxi.
- Every time we move **over** by one square we move **up** by 0.8 squares (notice that that's also the coefficient of x in the equation). This is the rate of charge of the taxi (cost per mile).
- If we move over by three squares, we move up by 3×0.8 squares.

Example 2

A small business has a debt of \$500,000 incurred from start-up costs. It predicts that it can pay off the debt at a rate of \$85,000 per year according to the following equation governing years in business (x) and debt measured in thousands of dollars (y).

$$y = -85x + 500$$

Graph the above equation and use your graph to predict when the debt will be fully paid.

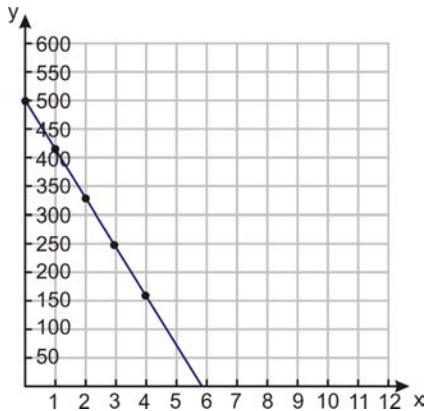
Solution

First, we start with our table of values:

Table 4.5:

x	y
0	500
1	415
2	330
3	245
4	160

Then we plot our points and draw the line that goes through them:



Notice the scale we've chosen here. There's no need to include any points above $y = 500$, but it's still wise to allow a little extra.

Next we need to determine how many years it takes the debt to reach zero, or in other words, what x -value will make the y -value equal 0. We know it's greater than four (since at $x = 4$ the y -value is still positive), so we need an x -scale that goes well past $x = 4$. Here we've chosen to show the x -values from 0 to 12, though there are many other places we could have chosen to stop.

To read the time that the debt is paid off, we simply read the point where the line hits $y = 0$ (the x -axis). It looks as if the line hits pretty close to $x = 6$. So **the debt will definitely be paid off in six years.**

To see more simple examples of graphing linear equations by hand, see the Khan Academy video on graphing lines at <http://www.youtube.com/watch?v=2UrcUfBizyw>. The narrator shows how to graph several linear equations, using a table of values to plot points and then connecting the points with a line.

Graphs and Equations of Horizontal and Vertical Lines

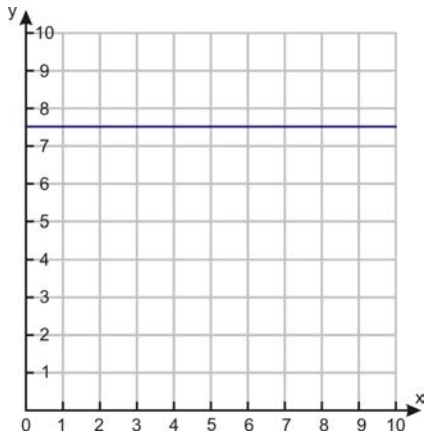
Example 3

"Mad-cabs" have an unusual offer going on. They are charging \$7.50 for a taxi ride of any length within the city limits. Graph the function that relates the cost of hiring the taxi (y) to the length of the journey in miles (x).

To proceed, the first thing we need is an **equation**. You can see from the problem that the cost of a journey doesn't depend on the length of the journey. It should come as no surprise that the equation then, does not have x in it. Since any value of x results in the same value of $y(7.5)$, the value you choose for x doesn't matter, so it isn't included in the equation. Here is the equation:

$$y = 7.5$$

The graph of this function is shown below. You can see that it's simply a horizontal line.



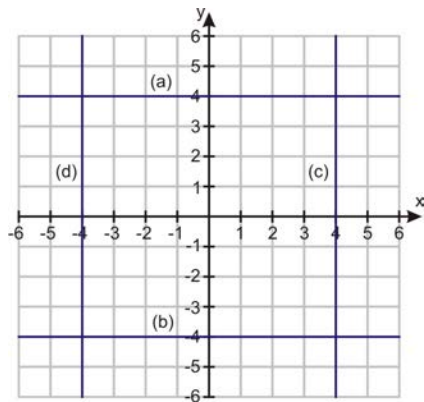
Any time you see an equation of the form “ $y = \text{constant}$,” the graph is a horizontal line that intercepts the y -axis at the value of the constant.

Similarly, when you see an equation of the form $x = \text{constant}$, then the graph is a vertical line that intercepts the x -axis at the value of the constant. (Notice that that kind of equation is a relation, and not a function, because each x -value (there’s only one in this case) corresponds to many (actually an infinite number) y -values.)

Example 4

Plot the following graphs.

- (a) $y = 4$
- (b) $y = -4$
- (c) $x = 4$
- (d) $x = -4$



- (a) $y = 4$ is a horizontal line that crosses the y -axis at 4.
- (b) $y = -4$ is a horizontal line that crosses the y -axis at -4 .
- (c) $x = 4$ is a vertical line that crosses the x -axis at 4.
- (d) $x = -4$ is a vertical line that crosses the x -axis at -4 .

Example 5

Find an equation for the x -axis and the y -axis.

Look at the axes on any of the graphs from previous examples. We have already said that they intersect at the origin (the point where $x = 0$ and $y = 0$). The following definition could easily work for each axis.

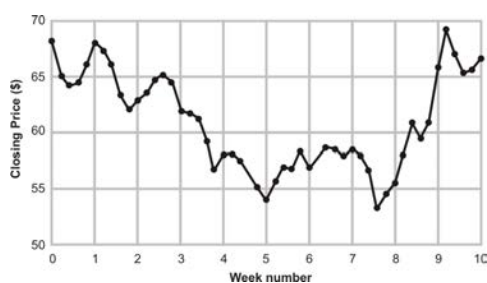
x -axis: A horizontal line crossing the y -axis at zero.

y -axis: A vertical line crossing the x -axis at zero.

So using example 3 as our guide, we could define the x -axis as the line $y = 0$ and the y -axis as the line $x = 0$.

Analyze Graphs of Linear Functions

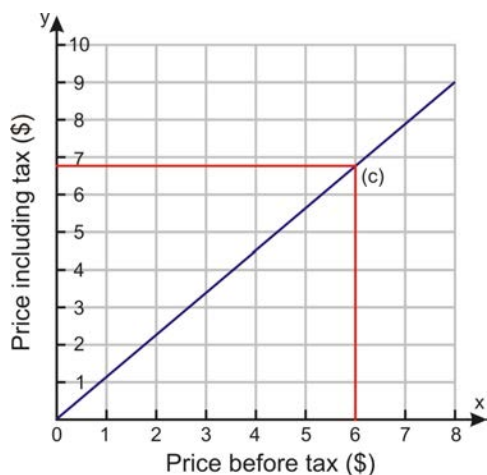
We often use graphs to represent relationships between two linked quantities. It's useful to be able to interpret the information that graphs convey. For example, the chart below shows a fluctuating stock price over ten weeks. You can read that the index closed the first week at about \$68, and at the end of the third week it was at about \$62. You may also see that in the first five weeks it lost about 20% of its value, and that it made about 20% gain between weeks seven and ten. Notice that this relationship is discrete, although the dots are connected to make the graph easier to interpret.



Analyzing graphs is a part of life - whether you are trying to decide to buy stock, figure out if your blog readership is increasing, or predict the temperature from a weather report. Many graphs are very complicated, so for now we'll start off with some simple linear conversion graphs. Algebra starts with basic relationships and builds to more complicated tasks, like reading the graph above.

Example 6

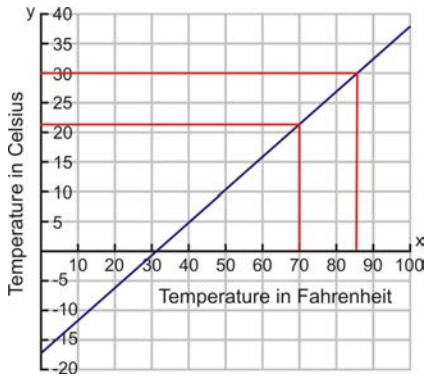
Below is a graph for converting marked prices in a downtown store into prices that include sales tax. Use the graph to determine the cost including sales tax for a \$6.00 pen in the store.



To find the relevant price with tax, first find the correct pre-tax price on the x -axis. This is the point $x = 6$.

Draw the line $x = 6$ up until it meets the function, then draw a horizontal line to the y -axis. This line hits at $y \approx 6.75$ (about three fourths of the way from $y = 6$ to $y = 7$).

The approximate cost including tax is \$6.75.



Example 7

The chart for converting temperature from Fahrenheit to Celsius is shown to the right. Use the graph to convert the following:

- 70° Fahrenheit to Celsius
- 0° Fahrenheit to Celsius
- 30° Celsius to Fahrenheit
- 0° Celsius to Fahrenheit

Solution

a) To find 70° Fahrenheit, we look along the Fahrenheit-axis (in other words the x -axis) and draw the line $x = 70$ up to the function. Then we draw a horizontal line to the Celsius-axis (y -axis). The horizontal line hits the axis at a little over 20 (21 or 22).

70° Fahrenheit is approximately equivalent to 21° Celsius.

b) To find 0° Fahrenheit, we just look at the y -axis. (Don't forget that this axis is simply the line $x = 0$.) The line hits the y -axis just below the half way point between -15 and -20 .

0° Fahrenheit is approximately equivalent to -18° Celsius.

c) To find 30° Celsius, we look up the Celsius-axis and draw the line $y = 30$ along to the function. When this horizontal line hits the function, we draw a line straight down to the Fahrenheit-axis. The line hits the axis at approximately 85.

30° Celsius is approximately equivalent to 85° Fahrenheit.

d) To find 0° Celsius, we look at the Fahrenheit-axis (the line $y = 0$). The function hits the x -axis just right of 30.

0° Celsius is equivalent to 32° Fahrenheit.

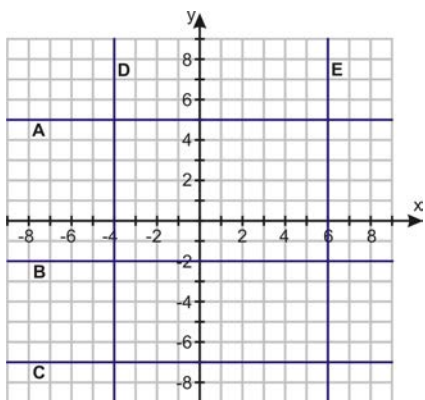
Lesson Summary

- Equations with the variables y and x can be graphed by making a chart of values that fit the equation and then plotting the values on a coordinate plane. This graph is simply another representation of the equation and can be analyzed to solve problems.
- Horizontal lines are defined by the equation $y = \text{constant}$ and vertical lines are defined by the equation $x = \text{constant}$.
- Be aware that although we graph the function as a line to make it easier to interpret, the function

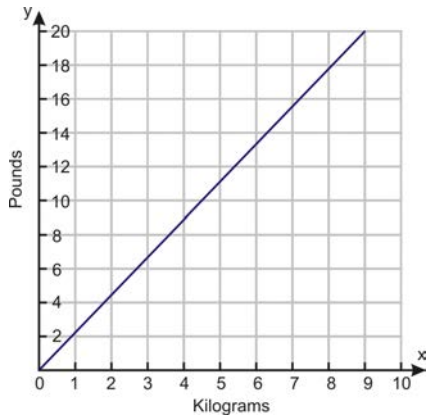
may actually be discrete.

Review Questions

1. Make a table of values for the following equations and then graph them.
 - (a) $y = 2x + 7$
 - (b) $y = 0.7x - 4$
 - (c) $y = 6 - 1.25x$
2. "Think of a number. Multiply it by 20, divide the answer by 9, and then subtract seven from the result."
 - (a) Make a table of values and plot the function that represents this sentence.
 - (b) If you picked 0 as your starting number, what number would you end up with?
 - (c) To end up with 12, what number would you have to start out with?
3. Write the equations for the five lines (*A* through *E*) plotted in the graph below.



4. In the graph above, at what points do the following lines intersect?
 - (a) *A* and *E*
 - (b) *A* and *D*
 - (c) *C* and *D*
 - (d) *B* and the *y*-axis
 - (e) *E* and the *x*-axis
 - (f) *C* and the line $y = x$
 - (g) *E* and the line $y = \frac{1}{2}x$
 - (h) *A* and the line $y = x + 3$
5. At the airport, you can change your money from dollars into euros. The service costs \$5, and for every additional dollar you get 0.7 euros.
 - (a) Make a table for this and plot the function on a graph.
 - (b) Use your graph to determine how many euros you would get if you give the office \$50.
 - (c) To get 35 euros, how many dollars would you have to pay?
 - (d) The exchange rate drops so that you can only get 0.5 euros per additional dollar. Now how many dollars do you have to pay for 35 euros?
6. The graph below shows a conversion chart for converting between weight in kilograms and weight in pounds. Use it to convert the following measurements.



- (a) 4 kilograms into weight in pounds
- (b) 9 kilograms into weight in pounds
- (c) 12 pounds into weight in kilograms
- (d) 17 pounds into weight in kilograms

7. Use the graph from problem 6 to answer the following questions.

- (a) An employee at a sporting goods store is packing 3-pound weights into a box that can hold 8 kilograms. How many weights can she place in the box?
- (b) After packing those weights, there is some extra space in the box that she wants to fill with one-pound weights. How many of those can she add?
- (c) After packing those, she realizes she misread the label and the box can actually hold 9 kilograms. How many more one-pound weights can she add?

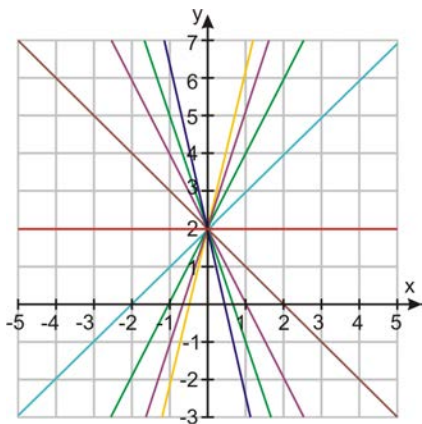
4.3 Graphing Using Intercepts

Learning Objectives

- Find intercepts of the graph of an equation.
- Use intercepts to graph an equation.
- Solve real-world problems using intercepts of a graph

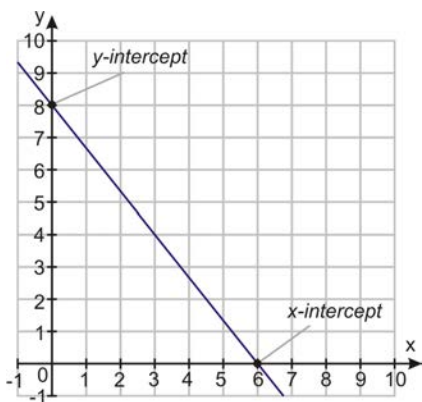
Introduction

Sanjit's office is 25 miles from home, and in traffic he expects the trip home to take him an hour if he starts at 5 PM. Today he hopes to stop at the post office along the way. If the post office is 6 miles from his office, when will Sanjit get there?



If you know just one of the points on a line, you'll find that isn't enough information to plot the line on a graph. As you can see in the graph above, there are many lines—in fact, infinitely many lines—that pass through a single point. But what if you know two points that are both on the line? Then there's only one way to graph that line; all you need to do is plot the two points and use a ruler to draw the line that passes through both of them.

There are a lot of options for choosing which two points on the line you use to plot it. In this lesson, we'll focus on two points that are rather convenient for graphing: the points where our line crosses the x - and y -axes, or **intercepts**. We'll see how to find intercepts algebraically and use them to quickly plot graphs.



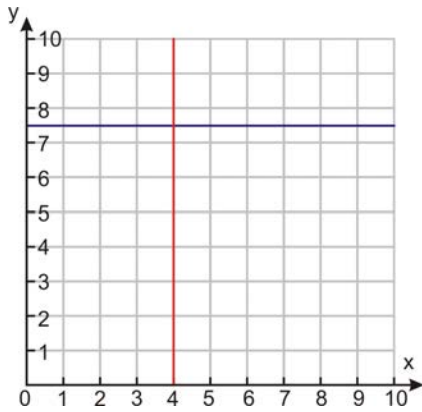
Look at the graph above. The **y -intercept** occurs at the point where the graph crosses the y -axis. The y -value at this point is 8, and the x -value is 0.

Similarly, the **x -intercept** occurs at the point where the graph crosses the x -axis. The x -value at this point is 6, and the y -value is 0.

So we know the coordinates of two points on the graph: $(0, 8)$ and $(6, 0)$. If we'd just been given those two coordinates out of the blue, we could quickly plot those points and join them with a line to recreate the above graph.

Note: Not all lines will have both an x - and a y -intercept, but most do. However, horizontal lines never cross the x -axis and vertical lines never cross the y -axis.

For examples of these special cases, see the graph below.



Finding Intercepts by Substitution

Example 1

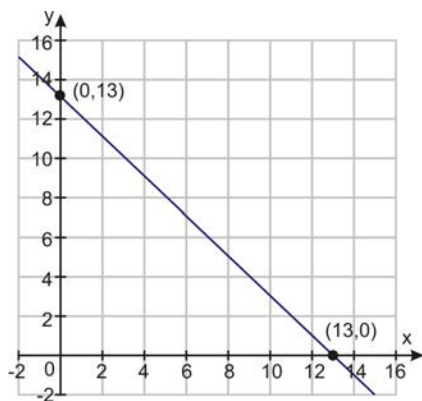
Find the intercepts of the line $y = 13 - x$ and use them to graph the function.

Solution

The first intercept is easy to find. The y -intercept occurs when $x = 0$. Substituting gives us $y = 13 - 0 = 13$, so the y -intercept is $(0, 13)$.

Similarly, the x -intercept occurs when $y = 0$. Plugging in 0 for y gives us $0 = 13 - x$, and adding x to both sides gives us $x = 13$. So $(13, 0)$ is the x -intercept.

To draw the graph, simply plot these points and join them with a line.



Example 2

Graph the following functions by finding intercepts.

- $y = 2x + 3$
- $y = 7 - 2x$
- $4x - 2y = 8$
- $2x + 3y = -6$

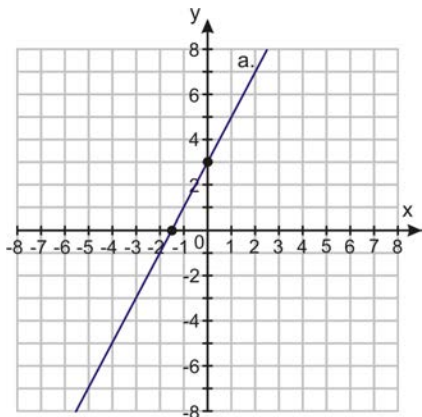
Solution

a) Find the y -intercept by plugging in $x = 0$:

$$y = 2 \cdot 0 + 3 = 3 \quad \text{– the } y \text{ – intercept is } (0, 3)$$

Find the x -intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 2x + 3 && \text{-- subtract 3 from both sides :} \\ -3 &= 2x && \text{-- divide by 2 :} \\ -\frac{3}{2} &= x && \text{-- the } x \text{ - intercept is } (-1.5, 0) \end{aligned}$$

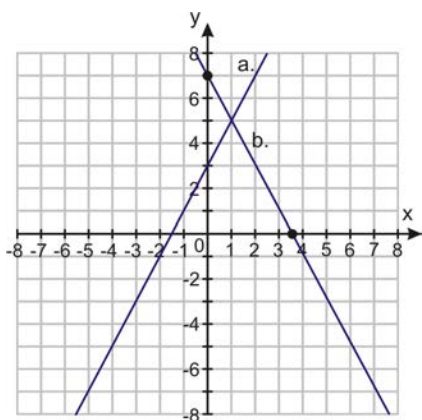


b) Find the y -intercept by plugging in $x = 0$:

$$y = 7 - 2 \cdot 0 = 7 \quad \text{-- the } y \text{ - intercept is } (0, 7)$$

Find the x -intercept by plugging in $y = 0$:

$$\begin{aligned} 0 &= 7 - 2x && \text{-- subtract 7 from both sides :} \\ -7 &= -2x && \text{-- divide by } -2 : \\ \frac{7}{2} &= x && \text{-- the } x \text{ - intercept is } (3.5, 0) \end{aligned}$$



c) Find the y -intercept by plugging in $x = 0$:

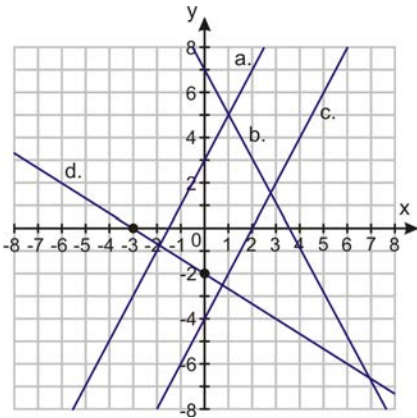
$$\begin{aligned} 4 \cdot 0 - 2y &= 8 \\ -2y &= 8 && \text{-- divide by } -2 \\ y &= -4 && \text{-- the } y \text{ - intercept is } (0, -4) \end{aligned}$$

Find the x -intercept by plugging in $y = 0$:

$$4x - 2 \cdot 0 = 8$$

$$4x = 8 \quad - \text{divide by } 4 :$$

$$x = 2 \quad - \text{the } x\text{-intercept is } (2, 0)$$



d) Find the y-intercept by plugging in $x = 0$:

$$2 \cdot 0 + 3y = -6$$

$$3y = -6 \quad - \text{divide by } 3 :$$

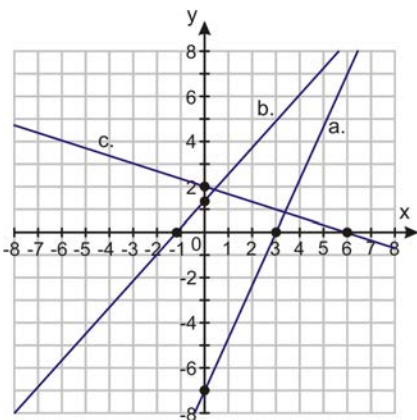
$$y = -2 \quad - \text{the } y\text{-intercept is } (0, -2)$$

Find the x-intercept by plugging in $y = 0$:

$$2x + 3 \cdot 0 = -6$$

$$2x = -6 \quad - \text{divide by } 2 :$$

$$x = -3 \quad - \text{the } x\text{-intercept is } (-3, 0)$$



Finding Intercepts for Standard Form Equations Using the Cover-Up Method

Look at the last two equations in example 2. These equations are written in *standard form*. Standard form equations are always written “**coefficient** times x plus (or minus) **coefficient** times y equals **value**”. In other words, they look like this:

$$ax + by = c$$

where a has to be positive, but b and c do not.

There is a neat method for finding intercepts in standard form, often referred to as the cover-up method.

Example 3

Find the intercepts of the following equations:

a) $7x - 3y = 21$


b) $12x - 10y = -15$

c) $x + 3y = 6$

Solution

To solve for each intercept, we realize that at the intercepts the value of **either** x **or** y is zero, and so any terms that contain that variable effectively drop out of the equation. To make a term disappear, simply cover it (a finger is an excellent way to cover up terms) and solve the resulting equation.

a) To solve for the y -intercept we set $x = 0$ and cover up the x -term:


$$-3y = 21$$

$$-3y = 21$$

$$y = -7 \quad (0, -7) \text{ is the } y\text{-intercept.}$$


Now we solve for the x -intercept:


$$7x = 21$$

$$7x = 21$$

$$x = 3 \quad (3, 0) \text{ is the } x\text{-intercept.}$$

b) To solve for the y -intercept ($x = 0$), cover up the x -term:


$$-10y = -15$$

$$-10y = -15$$

$$y = 1.5 \quad (0, 1.5) \text{ is the } y\text{-intercept.}$$


Now solve for the x -intercept ($y = 0$):


$$12x = -15$$

$$12x = -15$$

$$x = -\frac{5}{4} \quad (-1.25, 0) \text{ is the } x\text{-intercept.}$$

c) To solve for the y -intercept ($x = 0$), cover up the x -term:


$$3y = 6$$

$$3y = 6$$

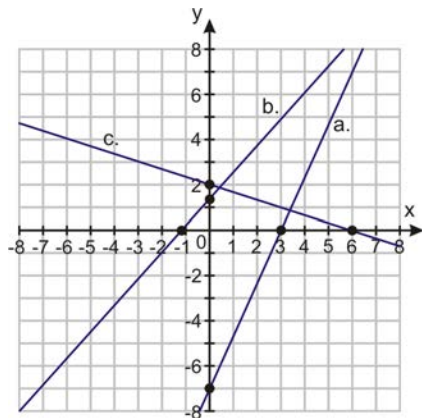
$y = 2$ (0, 2) is the y -intercept.

Solve for the y -intercept:

$$x \cdot 0 = 6$$

$x = 6$ (6, 0) is the x -intercept.

The graph of these functions and the intercepts is below:



To learn more about equations in standard form, try the Java applet at <http://www.analyze-math.com/line/line.htm> (scroll down and click the “click here to start” button.) You can use the sliders to change the values of a , b , and c and see how that affects the graph.

Solving Real-World Problems Using Intercepts of a Graph

Example 4

Jesus has \$30 to spend on food for a class barbecue. Hot dogs cost \$0.75 each (including the bun) and burgers cost \$1.25 (including the bun). Plot a graph that shows all the combinations of hot dogs and burgers he could buy for the barbecue, without spending more than \$30.

This time we will find an equation first, and then we can think logically about finding the intercepts.

If the number of burgers that Jesus buys is x , then the money he spends on burgers is $1.25x$

If the number of hot dogs he buys is y , then the money he spends on hot dogs is $0.75y$

So the total cost of the food is $1.25x + 0.75y$.

The total amount of money he has to spend is \$30, so if he is to spend it ALL, we can use the following equation:

$$1.25x + 0.75y = 30$$

We can solve for the intercepts using the cover-up method. First the y -intercept ($x = 0$):

$$0 + 0.75y = 30$$

$$0.75y = 30$$

$y = 40$ y -intercept: (0, 40)

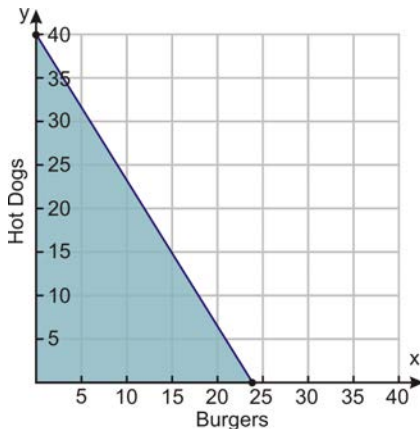
Then the x -intercept ($y = 0$):

$$1.25x + \text{🌭} = 30$$

$$1.25x = 30$$

$$x = 24 \quad x\text{-intercept: } (24, 0)$$

Now we plot those two points and join them to create our graph, shown here:



We could also have created this graph without needing to come up with an equation. We know that if John were to spend ALL the money on hot dogs, he could buy $\frac{30}{.75} = 40$ hot dogs. And if he were to buy only burgers he could buy $\frac{30}{1.25} = 24$ burgers. From those numbers, we can get 2 intercepts: (0 burgers, 40 hot dogs) and (24 burgers, 0 hot dogs). We could plot these just as we did above and obtain our graph that way.

As a final note, we should realize that Jesus' problem is really an example of an inequality. He can, in fact, spend any amount up to \$30. The only thing he cannot do is spend more than \$30. The graph above reflects this: the line is the set of solutions that involve spending exactly \$30, and the shaded region shows solutions that involve spending less than \$30. We'll work with inequalities some more in Chapter 6.

Lesson Summary

- A **y-intercept** occurs at the point where a graph crosses the y -axis (where $x = 0$) and an **x-intercept** occurs at the point where a graph crosses the x -axis (where $y = 0$).
- The y -intercept can be found by **substituting** $x = 0$ into the equation and solving for y . Likewise, the x -intercept can be found by **substituting** $y = 0$ into the equation and solving for x .
- A linear equation is in **standard form** if it is written as "positive coefficient times x plus coefficient times y equals value". Equations in standard form can be solved for the intercepts by covering up the x (or y) term and solving the equation that remains.

Review Questions

1. Find the intercepts for the following equations using substitution.
 - (a) $y = 3x - 6$
 - (b) $y = -2x + 4$
 - (c) $y = 14x - 21$
 - (d) $y = 7 - 3x$

- (e) $y = 2.5x - 4$
 (f) $y = 1.1x + 2.2$
 (g) $y = \frac{3}{8}x + 7$
 (h) $y = \frac{5}{9} - \frac{2}{7}x$
2. Find the intercepts of the following equations using the cover-up method.
- (a) $5x - 6y = 15$
 (b) $3x - 4y = -5$
 (c) $2x + 7y = -11$
 (d) $5x + 10y = 25$
 (e) $5x - 1.3y = 12$
 (f) $1.4x - 3.5y = 7$
 (g) $\frac{3}{5}x + 2y = \frac{2}{5}$
 (h) $\frac{3}{4}x - \frac{2}{3}y = \frac{1}{5}$
3. Use any method to find the intercepts and then graph the following equations.
- (a) $y = 2x + 3$
 (b) $6(x - 1) = 2(y + 3)$
 (c) $x - y = 5$
 (d) $x + y = 8$
4. At the local grocery store strawberries cost \$3.00 per pound and bananas cost \$1.00 per pound.
- (a) If I have \$10 to spend on strawberries and bananas, draw a graph to show what combinations of each I can buy and spend exactly \$10.
 (b) Plot the point representing 3 pounds of strawberries and 2 pounds of bananas. Will that cost more or less than \$10?
 (c) Do the same for the point representing 1 pound of strawberries and 5 pounds of bananas.
5. A movie theater charges \$7.50 for adult tickets and \$4.50 for children. If the theater takes in \$900 in ticket sales for a particular screening, draw a graph which depicts the possibilities for the number of adult tickets and the number of child tickets sold.
6. Why can't we use the intercept method to graph the following equation? $3(x + 2) = 2(y + 3)$
7. Name two more equations that we can't use the intercept method to graph.

4.4 Slope and Rate of Change

Learning Objectives

- Find positive and negative slopes.
- Recognize and find slopes for horizontal and vertical lines.
- Understand rates of change.
- Interpret graphs and compare rates of change.

Introduction

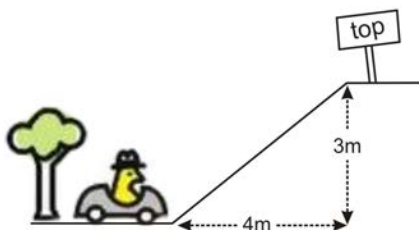
Wheelchair ramps at building entrances must have a slope between $\frac{1}{16}$ and $\frac{1}{20}$. If the entrance to a new office building is 28 inches off the ground, how long does the wheelchair ramp need to be?

We come across many examples of slope in everyday life. For example, a slope is in the pitch of a roof, the grade or incline of a road, or the slant of a ladder leaning on a wall. In math, we use the word **slope** to define steepness in a particular way.

$$\text{Slope} = \frac{\text{distance moved vertically}}{\text{distance moved horizontally}}$$

To make it easier to remember, we often word it like this:

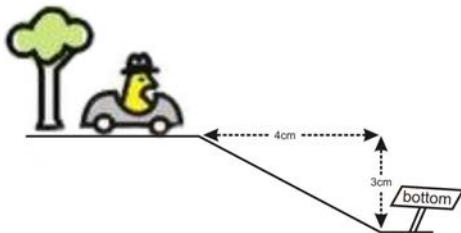
$$\text{Slope} = \frac{\text{rise}}{\text{run}}$$



In the picture above, the slope would be the ratio of the **height** of the hill to the horizontal **length** of the hill. In other words, it would be $\frac{3}{4}$, or 0.75.

If the car were driving to the **right** it would **climb** the hill - we say this is a positive slope. Any time you see the graph of a line that goes up as you move to the right, the slope is **positive**.

If the car kept driving after it reached the top of the hill, it might go down the other side. If the car is driving to the **right** and **descending**, then we would say that the slope is **negative**.



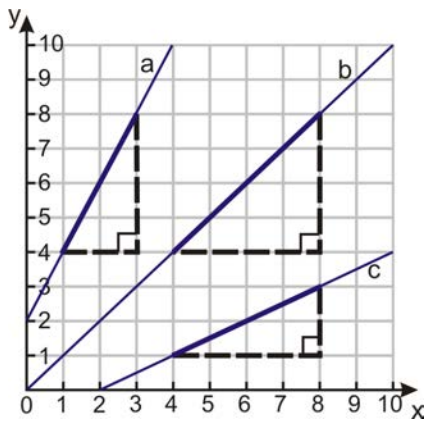
Here's where it gets tricky: If the car turned around instead and drove back down the left side of the hill, the slope of that side would still be positive. This is because the rise would be -3 , but the run would be -4 (think of the x -axis - if you move from right to left you are moving in the **negative** x -direction). That means our slope ratio would be $\frac{-3}{-4}$, and the negatives cancel out to leave 0.75, the same slope as before. In other words, the slope of a line is the same no matter which direction you travel along it.

Find the Slope of a Line

A simple way to find a value for the slope of a line is to draw a right triangle whose hypotenuse runs along the line. Then we just need to measure the distances on the triangle that correspond to the rise (the vertical dimension) and the run (the horizontal dimension).

Example 1

Find the slopes for the three graphs shown.



Solution

There are already right triangles drawn for each of the lines - in future problems you'll do this part yourself. Note that it is easiest to make triangles whose vertices are **lattice points** (i.e. points whose coordinates are all integers).

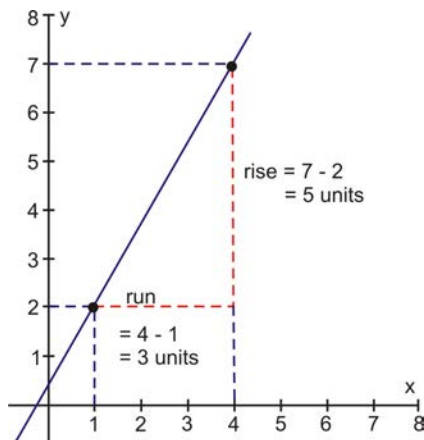
- a) The rise shown in this triangle is 4 units; the run is 2 units. The slope is $\frac{4}{2} = 2$.
- b) The rise shown in this triangle is 4 units, and the run is also 4 units. The slope is $\frac{4}{4} = 1$.
- c) The rise shown in this triangle is 2 units, and the run is 4 units. The slope is $\frac{2}{4} = \frac{1}{2}$.

Example 2

Find the slope of the line that passes through the points $(1, 2)$ and $(4, 7)$.

Solution

We already know how to graph a line if we're given two points: we simply plot the points and connect them with a line. Here's the graph:



Since we already have coordinates for the vertices of our right triangle, we can quickly work out that the rise is $7 - 2 = 5$ and the run is $4 - 1 = 3$ (see diagram). So the slope is $\frac{7-2}{4-1} = \frac{5}{3}$.

If you look again at the calculations for the slope, you'll notice that the 7 and 2 are the y -coordinates of the two points and the 4 and 1 are the x -coordinates. This suggests a pattern we can follow to get a general formula for the slope between two points (x_1, y_1) and (x_2, y_2) :

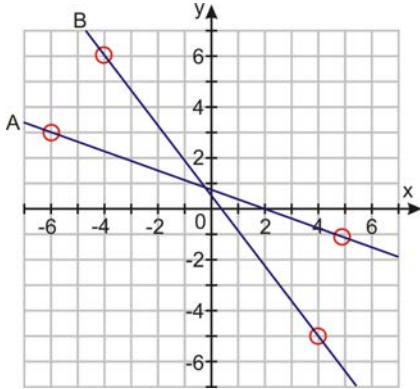
$$\text{Slope between } (x_1, y_1) \text{ and } (x_2, y_2) = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\text{or } m = \frac{\Delta y}{\Delta x}$$

In the second equation the letter m denotes the slope (this is a mathematical convention you'll see often) and the Greek letter delta (Δ) means **change**. So another way to express slope is *change in y* divided by *change in x*. In the next section, you'll see that it doesn't matter which point you choose as point 1 and which you choose as point 2.

Example 3

Find the slopes of the lines on the graph below.



Solution

Look at the lines - they both slant down (or decrease) as we move from left to right. Both these lines have **negative slope**.

The lines don't pass through very many convenient lattice points, but by looking carefully you can see a few points that look to have integer coordinates. These points have been circled on the graph, and we'll use them to determine the slope. We'll also do our calculations twice, to show that we get the same slope whichever way we choose point 1 and point 2.

For Line A:

$$(x_1, y_1) = (-6, 3) \quad (x_2, y_2) = (5, -1)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-1) - (3)}{(5) - (-6)} = \frac{-4}{11} \approx -0.364$$

$$(x_1, y_1) = (5, -1) \quad (x_2, y_2) = (-6, 3)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (-1)}{(-6) - (5)} = \frac{4}{-11} \approx -0.364$$

For Line B

$$(x_1, y_1) = (-4, 6) \quad (x_2, y_2) = (4, -5)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(-5) - (6)}{(4) - (-4)} = \frac{-11}{8} = -1.375$$

$$(x_1, y_1) = (4, -5) \quad (x_2, y_2) = (-4, 6)$$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(6) - (-5)}{(-4) - (4)} = \frac{11}{-8} = -1.375$$

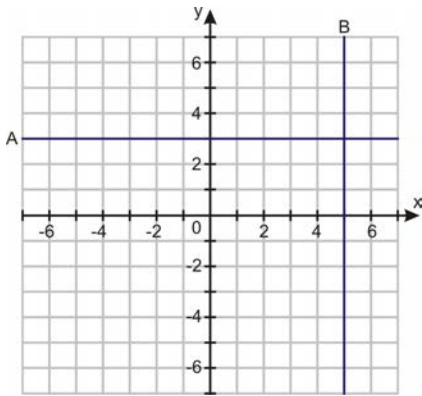
You can see that whichever way round you pick the points, the answers are the same. Either way, **Line A has slope -0.364, and Line B has slope -1.375.**

Khan Academy has a series of videos on finding the slope of a line, starting at <http://www.youtube.com/watch?v=hXP1Gv9IMBo>.

Find the Slopes of Horizontal and Vertical lines

Example 4

Determine the slopes of the two lines on the graph below.



Solution

There are 2 lines on the graph: $A(y = 3)$ and $B(x = 5)$.

Let's pick 2 points on line A —say, $(x_1, y_1) = (-4, 3)$ and $(x_2, y_2) = (5, 3)$ —and use our equation for slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(3) - (3)}{(5) - (-4)} = \frac{0}{9} = 0.$$

If you think about it, this makes sense - if y doesn't change as x increases then there is no slope, or rather, the slope is zero. You can see that this must be true for all horizontal lines.

Horizontal lines ($y = \text{constant}$) all have a slope of 0.

Now let's consider line B . If we pick the points $(x_1, y_1) = (5, -3)$ and $(x_2, y_2) = (5, 4)$, our slope equation is $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(4) - (-3)}{(5) - (5)} = \frac{7}{0}$. But dividing by zero isn't allowed!

In math we often say that a term which involves division by zero is **undefined**. (Technically, the answer can also be said to be infinitely large—or infinitely small, depending on the problem.)

Vertical lines ($x = \text{constant}$) all have an infinite (or undefined) slope.

Find a Rate of Change

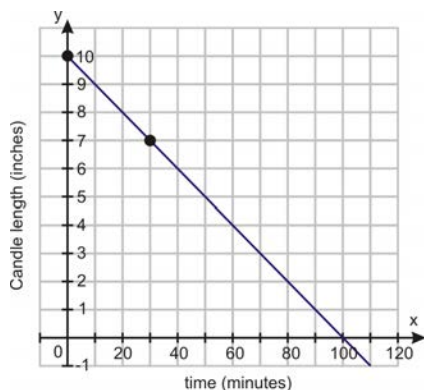
The slope of a function that describes real, measurable quantities is often called a **rate of change**. In that case the slope refers to a change in one quantity (y) **per** unit change in another quantity (x). (This is where the equation $m = \frac{\Delta y}{\Delta x}$ comes in—remember that Δy and Δx represent the change in y and x respectively.)

Example 5

A candle has a starting length of 10 inches. 30 minutes after lighting it, the length is 7 inches. Determine the rate of change in length of the candle as it burns. Determine how long the candle takes to completely burn to nothing.

Solution

First we'll graph the function to visualize what is happening. We have 2 points to start with: we know that at the moment the candle is lit ($\text{time} = 0$) the length of the candle is 10 inches, and after 30 minutes ($\text{time} = 30$) the length is 7 inches. Since the candle length depends on the time, we'll plot time on the horizontal axis, and candle length on the vertical axis.



The rate of change of the candle's length is simply the slope of the line. Since we have our 2 points $(x_1, y_1) = (0, 10)$ and $(x_2, y_2) = (30, 7)$, we can use the familiar version of the slope formula:

$$\text{Rate of change} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(7 \text{ inches}) - (10 \text{ inches})}{(30 \text{ minutes}) - (0 \text{ minutes})} = \frac{-3 \text{ inches}}{30 \text{ minutes}} = -0.1 \text{ inches per minute}$$

Note that the slope is negative. A negative rate of change means that the quantity is decreasing with time—just as we would expect the length of a burning candle to do.

To find the point when the candle reaches zero length, we can simply read the x -intercept off the graph (100 minutes). We can use the rate equation to verify this algebraically:

$$\begin{aligned} \text{Length burned} &= \text{rate} \times \text{time} \\ 10 &= 0.1 \times 100 \end{aligned}$$

Since the candle length was originally 10 inches, our equation confirms that 100 minutes is the time taken.

Example 6

The population of fish in a certain lake increased from 370 to 420 over the months of March and April. At what rate is the population increasing?

Solution

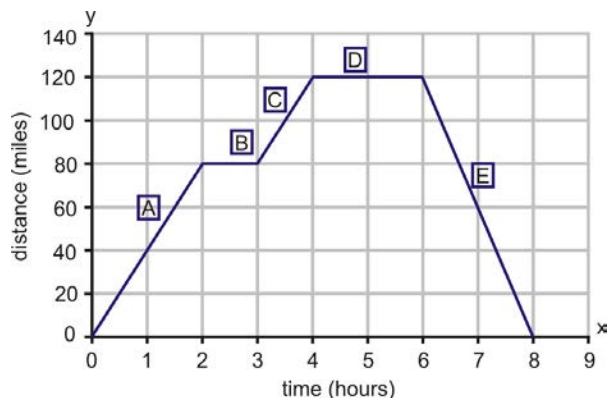
Here we don't have two points from which we can get x - and y -coordinates for the slope formula. Instead, we'll need to use the alternate formula, $m = \frac{\Delta y}{\Delta x}$.

The change in y -values, or Δy , is the change in the number of fish, which is $420 - 370 = 50$. The change in x -values, Δx , is the amount of time over which this change took place: two months. So $\frac{\Delta y}{\Delta x} = \frac{50 \text{ fish}}{2 \text{ months}}$, or **25 fish per month**.

Interpret a Graph to Compare Rates of Change

Example 7

The graph below represents a trip made by a large delivery truck on a particular day. During the day the truck made two deliveries, one taking an hour and the other taking two hours. Identify what is happening at each stage of the trip (stages A through E).



Solution

Here are the stages of the trip:

- The truck sets off and travels 80 miles in 2 hours.
- The truck covers no distance for 2 hours.
- The truck covers $(120 - 80) = 40$ miles in 1 hour.
- The truck covers no distance for 1 hour.
- The truck covers -120 miles in 2 hours.

Let's look at each section more closely.

A. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{80 \text{ miles}}{2 \text{ hours}} = 40$ miles per hour

Notice that the rate of change is a **speed**—or rather, a **velocity**. (The difference between the two is that velocity has a direction, and speed does not. In other words, velocity can be either positive or negative, with negative velocity representing travel in the opposite direction. You'll see the difference more clearly in part E.)

Since velocity equals distance divided by time, the slope (or rate of change) of a distance-time graph is always a velocity.

So during the first part of the trip, the truck travels at a constant speed of 40 mph for 2 hours, covering a distance of 80 miles.

B. The slope here is 0, so the rate of change is 0 mph. The truck is stationary for one hour. This is the first delivery stop.

C. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(120-80) \text{ miles}}{(4-3) \text{ hours}} = 40$ miles per hour. The truck is traveling at 40 mph.

D. Once again the slope is 0, so the rate of change is 0 mph. The truck is stationary for two hours. This is the second delivery stop. At this point the truck is 120 miles from the start position.

E. Rate of change = $\frac{\Delta y}{\Delta x} = \frac{(0-120) \text{ miles}}{(8-6) \text{ hours}} = \frac{-120 \text{ miles}}{2 \text{ hours}} = -60$ miles per hour. The truck is traveling at *negative* 60 mph.

Wait – a negative speed? Does that mean that the truck is reversing? Well, probably not. It's actually the *velocity* and not the speed that is negative, and a negative velocity simply means that the distance *from the starting position* is decreasing with time. The truck is driving in the opposite direction – back to where it started from. Since it no longer has 2 heavy loads, it travels faster (60 mph instead of 40 mph), covering the 120 mile return trip in 2 hours. Its *speed* is 60 mph, and its *velocity* is -60 mph, because it is traveling in the opposite direction from when it started out.

Lesson Summary

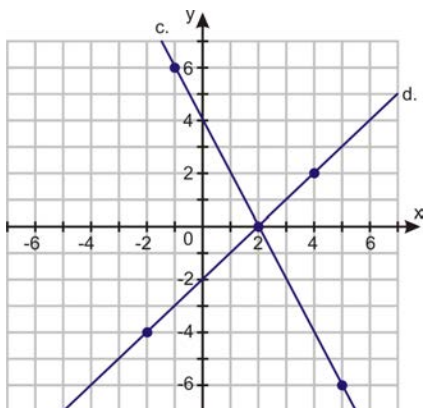
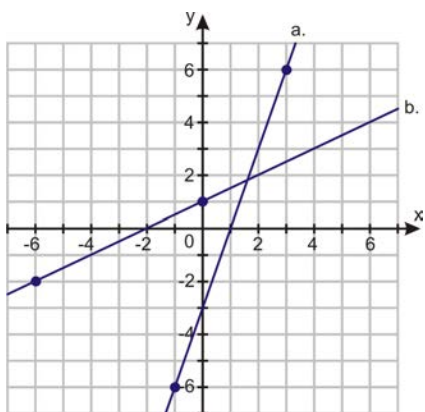
- **Slope** is a measure of change in the vertical direction for each step in the horizontal direction. Slope is often represented as “ m ”.
- Slope can be expressed as $\frac{\text{rise}}{\text{run}}$, or $\frac{\Delta y}{\Delta x}$.
- The slope between two points (x_1, y_1) and (x_2, y_2) is equal to $\frac{y_2 - y_1}{x_2 - x_1}$.
- **Horizontal lines** (where $y = a$ constant) all have a slope of 0.
- **Vertical lines** (where $x = a$ constant) all have an infinite (or undefined) slope.
- The slope (or **rate of change**) of a distance-time graph is a **velocity**.

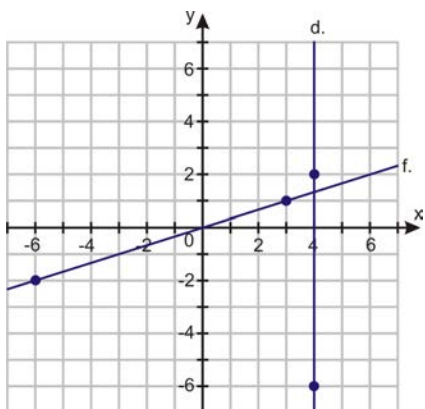
Review Questions

1. Use the slope formula to find the slope of the line that passes through each pair of points.

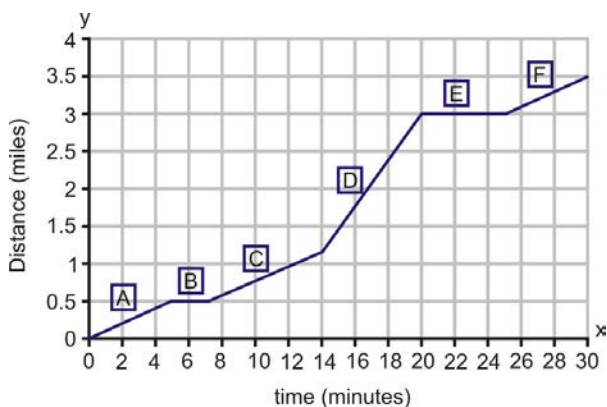
- (a) $(-5, 7)$ and $(0, 0)$
- (b) $(-3, -5)$ and $(3, 11)$
- (c) $(3, -5)$ and $(-2, 9)$
- (d) $(-5, 7)$ and $(-5, 11)$
- (e) $(9, 9)$ and $(-9, -9)$
- (f) $(3, 5)$ and $(-2, 7)$
- (g) $(2.5, 3)$ and $(8, 3.5)$

2. For each line in the graphs below, use the points indicated to determine the slope.





- For each line in the graphs above, imagine another line with the same slope that passes through the point (1, 1), and name one more point on that line.
- The graph below is a distance-time graph for Mark's three and a half mile cycle ride to school. During this ride, he rode on cycle paths but the terrain was hilly. He rode slower up hills and faster down them. He stopped once at a traffic light and at one point he stopped to mend a punctured tire. The graph shows his distance from home at any given time. Identify each section of the graph accordingly.



4.5 Graphs Using Slope-Intercept Form

Learning Objectives

- Identify the slope and y -intercept of equations and graphs.
- Graph an equation in slope-intercept form.
- Understand what happens when you change the slope or intercept of a line.
- Identify parallel lines from their equations.

Introduction

The total profit of a business is described by the equation $y = 15000x - 80000$, where x is the number of months the business has been running. How much profit is the business making per month, and what were its start-up costs? How much profit will it have made in a year?

Identify Slope and y-intercept

So far, we've been writing a lot of our equations in **slope-intercept form**—that is, we've been writing them in the form $y = mx + b$, where m and b are both constants. It just so happens that m is the slope and the point $(0, b)$ is the y-intercept of the graph of the equation, which gives us enough information to draw the graph quickly.

Example 1

Identify the slope and y-intercept of the following equations.

a) $y = 3x + 2$

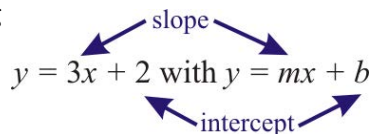
b) $y = 0.5x - 3$

c) $y = -7x$

d) $y = -4$

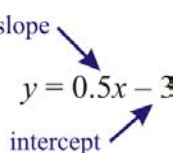
Solution

a) Comparing $y = 3x + 2$ with $y = mx + b$, we can see that $m = 3$ and $b = 2$. So $y = 3x + 2$ has a **slope**



of **3** and a y-intercept of **(0, 2)**.

b) $y = 0.5x - 3$ has a **slope of 0.5** and a y-intercept of **(0, -3)**.



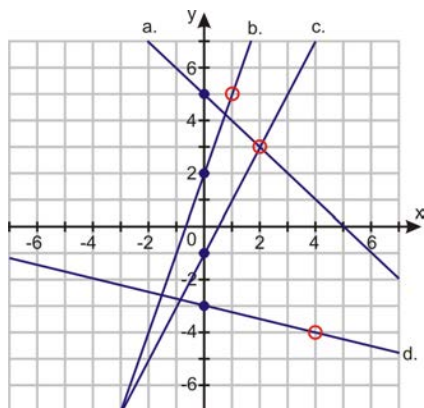
Notice that the intercept is **negative**. The b -term includes the sign of the operator (plus or minus) in front of the number—for example, $y = 0.5x - 3$ is identical to $y = 0.5x + (-3)$, and that means that b is -3 , not just 3.

c) At first glance, this equation doesn't look like it's in slope-intercept form. But we can rewrite it as $y = -7x + 0$, and that means it has a **slope of -7** and a y-intercept of **(0, 0)**. Notice that the slope is negative and the line passes through the origin.

d) We can rewrite this one as $y = 0x - 4$, giving us a **slope of 0** and a y-intercept of **(0, -4)**. This is a horizontal line.

Example 2

Identify the slope and y-intercept of the lines on the graph shown below.



The intercepts have been marked, as well as some convenient lattice points that the lines pass through.

Solution

- a) **The y-intercept is (0, 5).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-2}{2} = -1$.
- b) **The y-intercept is (0, 2).** The line also passes through (1, 5), so the slope is $\frac{\Delta y}{\Delta x} = \frac{3}{1} = 3$.
- c) **The y-intercept is (0, -1).** The line also passes through (2, 3), so the slope is $\frac{\Delta y}{\Delta x} = \frac{4}{2} = 2$.
- d) **The y-intercept is (0, -3).** The line also passes through (4, -4), so the slope is $\frac{\Delta y}{\Delta x} = \frac{-1}{4} = -\frac{1}{4}$ or -0.25.

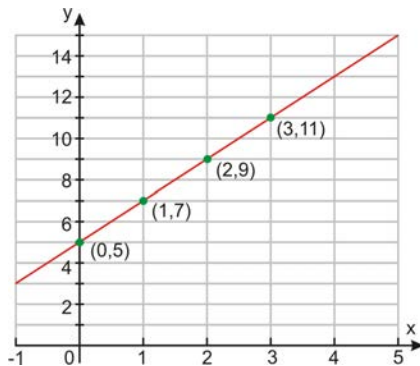
Graph an Equation in Slope-Intercept Form

Once we know the slope and intercept of a line, it's easy to graph it. Just remember what slope means. Let's look back at this example from Lesson 4.1.

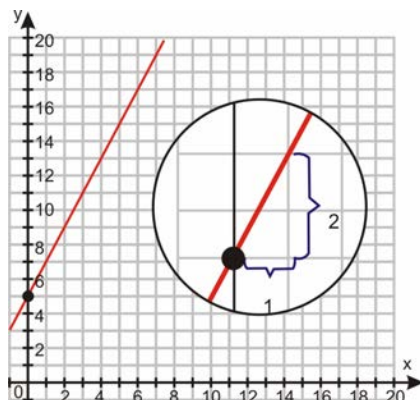
Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter x to stand for the number he thought of and the letter y to represent the final result of applying the rule. He wrote his rule in the form of an equation: $y = 2x + 5$.

Help him visualize what is going on by graphing the function that this rule describes.

In that example, we constructed a table of values, and used that table to plot some points to create our graph.



We also saw another way to graph this equation. Just by looking at the equation, we could see that the y-intercept was (0, 5), so we could start by plotting that point. Then we could also see that the slope was 2, so we could find another point on the graph by going over 1 unit and up 2 units. The graph would then be the line between those two points.



Here's another problem where we can use the same method.

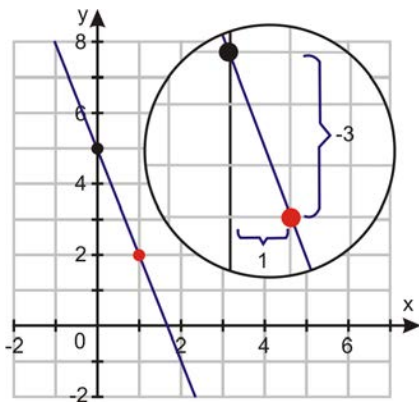
Example 3

Graph the following function: $y = -3x + 5$

Solution

To graph the function without making a table, follow these steps:

1. Identify the y -intercept: $b = 5$
2. Plot the intercept: $(0, 5)$
3. Identify the slope: $m = -3$. (This is equal to $\frac{-3}{1}$, so the **rise** is -3 and the **run** is 1 .)
4. Move **over** 1 unit and **down** 3 units to find another point on the line: $(1, 2)$
5. Draw the line through the points $(0, 5)$ and $(1, 2)$.



Notice that to graph this equation based on its slope, we had to find the rise and run—and it was easiest to do that when the slope was expressed as a fraction. That's true in general: to graph a line with a particular slope, it's easiest to first express the slope as a fraction in simplest form, and then read off the numerator and the denominator of the fraction to get the rise and run of the graph.

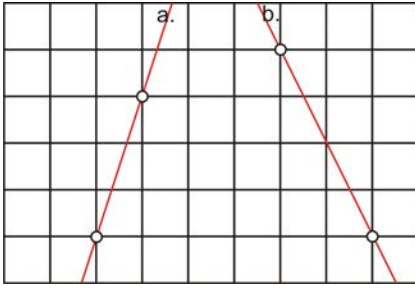
Example 4

Find integer values for the **rise** and **run** of the following slopes, then graph lines with corresponding slopes.

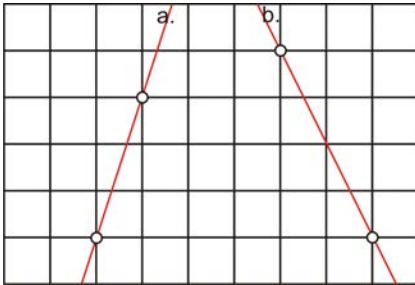
- a) $m = 3$
- b) $m = -2$
- c) $m = 0.75$
- d) $m = -0.375$

Solution

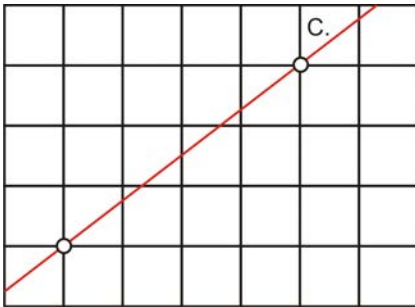
- a)
- $3 = \frac{3}{1}$ As we move **across** 1 unit we move **up** by 3
-



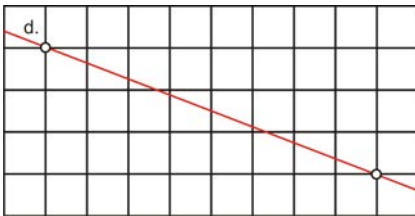
b) $-2 = \frac{-2}{1}$ As we move **across** 1 unit we move **down** by 2



c) $0.75 = \frac{3}{4}$ As we move **across** 4 units we move **up** by 3

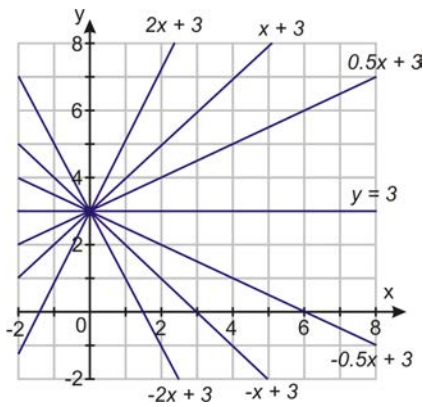


d) $-0.375 = \frac{-3}{8}$ As we move **across** 8 units we move **down** by 3



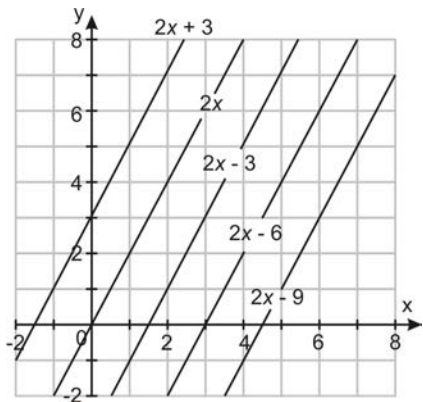
Changing the Slope or Intercept of a Line

The following graph shows a number of lines with different slopes, but all with the same y-intercept: (0, 3).



You can see that all the functions with positive slopes increase as we move from left to right, while all functions with negative slopes decrease as we move from left to right. Another thing to notice is that the greater the slope, the steeper the graph.

This graph shows a number of lines with the same slope, but different y -intercepts.



Notice that changing the intercept simply translates (shifts) the graph up or down. Take a point on the graph of $y = 2x$, such as $(1, 2)$. The corresponding point on $y = 2x + 3$ would be $(1, 5)$. Adding 3 to the y -intercept means we also add 3 to every other y -value on the graph. Similarly, the corresponding point on the $y = 2x - 3$ line would be $(1, -1)$; we would subtract 3 from the y -value and from every other y -value.

Notice also that these lines all appear to be parallel. Are they truly parallel?

To answer that question, we'll use a technique that you'll learn more about in a later chapter. We'll take 2 of the equations—say, $y = 2x$ and $y = 2x + 3$ —and solve for values of x and y that satisfy both equations. That will tell us at what point those two lines intersect, if any. (Remember that **parallel lines**, by definition, are lines that don't intersect.)

So what values would satisfy both $y = 2x$ and $y = 2x + 3$? Well, if both of those equations were true, then y would be equal to both $2x$ and $2x + 3$, which means those two expressions would also be equal to each other. So we can get our answer by solving the equation $2x = 2x + 3$.

But what happens when we try to solve that equation? If we subtract $2x$ from both sides, we end up with $0 = 3$. That can't be true no matter what x equals. And that means that there just isn't any value for x that will make both of the equations we started out with true. In other words, there isn't any point where those two lines intersect. They are parallel, just as we thought.

And we'd find out the same thing no matter which two lines we'd chosen. In general, since changing the intercept of a line just results in shifting the graph up or down, the new line will always be parallel to the old line as long as the slope stays the same.

Any two lines with identical slopes are parallel.

Further Practice

To get a better understanding of what happens when you change the slope or the y-intercept of a linear equation, try playing with the Java applet at <http://standards.nctm.org/document/eexamples/chap7/7.5/index.htm>.

Lesson Summary

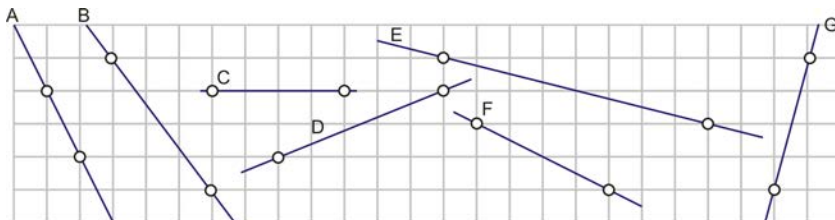
- A common form of a line (linear equation) is **slope-intercept form**: $y = mx + b$, where m is the slope and the point $(0, b)$ is the y-intercept
- Graphing a line in slope-intercept form is a matter of first plotting the y-intercept $(0, b)$, then finding a second point based on the slope, and using those two points to graph the line.
- Any two lines with identical slopes are **parallel**.

Review Questions

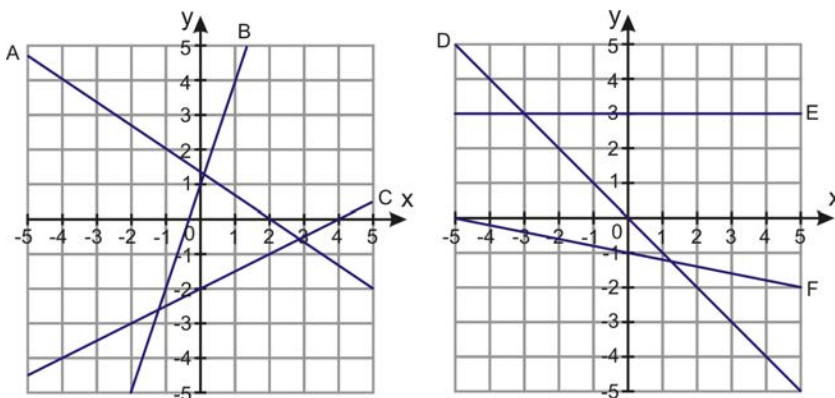
1. Identify the slope and y-intercept for the following equations.

- (a) $y = 2x + 5$
- (b) $y = -0.2x + 7$
- (c) $y = x$
- (d) $y = 3.75$

2. Identify the slope of the following lines.



3. Identify the slope and y-intercept for the following functions.



4. Plot the following functions on a graph.

- (a) $y = 2x + 5$
- (b) $y = -0.2x + 7$
- (c) $y = x$

- (d) $y = 3.75$
5. Which two of the following lines are parallel?
- (a) $y = 2x + 5$
 - (b) $y = -0.2x + 7$
 - (c) $y = x$
 - (d) $y = 3.75$
 - (e) $y = -\frac{1}{5}x - 11$
 - (f) $y = -5x + 5$
 - (g) $y = -3x + 11$
 - (h) $y = 3x + 3.5$
6. What is the y -intercept of the line passing through $(1, -4)$ and $(3, 2)$?
7. What is the y -intercept of the line with slope -2 that passes through $(3, 1)$?
8. Line A passes through the points $(2, 6)$ and $(-4, 3)$. Line B passes through the point $(3, 2.5)$, and is parallel to line A
- (a) Write an equation for line A in slope-intercept form.
 - (b) Write an equation for line B in slope-intercept form.
9. Line C passes through the points $(2, 5)$ and $(1, 3.5)$. Line D is parallel to line C , and passes through the point $(2, 6)$. Name another point on line D . (Hint: you can do this without graphing or finding an equation for either line.)

4.6 Direct Variation Models

Learning Objectives

- Identify direct variation.
- Graph direct variation equations.
- Solve real-world problems using direct variation models.

Introduction

Suppose you see someone buy five pounds of strawberries at the grocery store. The clerk weighs the strawberries and charges \$12.50 for them. Now suppose you wanted two pounds of strawberries for yourself. How much would you expect to pay for them?

Identify Direct Variation

The preceding problem is an example of a **direct variation**. We would expect that the strawberries are priced on a “per pound” basis, and that if you buy two-fifths the amount of strawberries, you would pay two-fifths of \$12.50 for your strawberries, or \$5.00.

Similarly, if you bought 10 pounds of strawberries (twice the amount) you would pay twice \$12.50, and if you did not buy any strawberries you would pay nothing.

If variable y varies directly with variable x , then we write the relationship as $y = k \cdot x$. k is called the **constant of proportionality**.

If we were to graph this function, you can see that it would pass through the origin, because $y = 0$ when $x = 0$, whatever the value of k . So we know that a direct variation, when graphed, has a single intercept at $(0, 0)$.

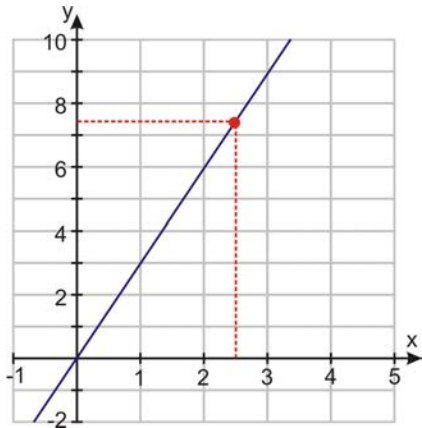
Example 1

If y varies directly with x according to the relationship $y = k \cdot x$, and $y = 7.5$ when $x = 2.5$, determine the constant of proportionality, k .

Solution

We can solve for the constant of proportionality using substitution. Substitute $x = 2.5$ and $y = 7.5$ into the equation $y = k \cdot x$ to get $7.5 = k(2.5)$. Then divide both sides by 2.5 to get $k = \frac{7.5}{2.5} = 3$. **The constant of proportionality, k , is 3.**

We can graph the relationship quickly, using the intercept $(0, 0)$ and the point $(2.5, 7.5)$. The graph is shown below. It is a straight line with slope 3.



The graph of a direct variation always passes through the origin, and always has a slope that is equal to the constant of proportionality, k .

Example 2

The volume of water in a fish-tank, V , varies directly with depth, d . If there are 15 gallons in the tank when the depth is 8 inches, calculate how much water is in the tank when the depth is 20 inches.

Solution

This is a good example of a direct variation, but for this problem we'll have to determine the equation of the variation ourselves. Since the volume, V , depends on depth, d , we'll use an equation of the form $y = k \cdot x$, but in place of y we'll use V and in place of x we'll use d :

$$V = k \cdot d$$

We know that when the depth is 8 inches the volume is 15 gallons, so to solve for k , we plug in 15 for V and 8 for d to get $15 = k(8)$. Dividing by 8 gives us $k = \frac{15}{8} = 1.875$.

Now to find the volume of water at the final depth, we use $V = k \cdot d$ again, but this time we can plug in our new d and the value we found for k :

$$V = 1.875 \times 20$$

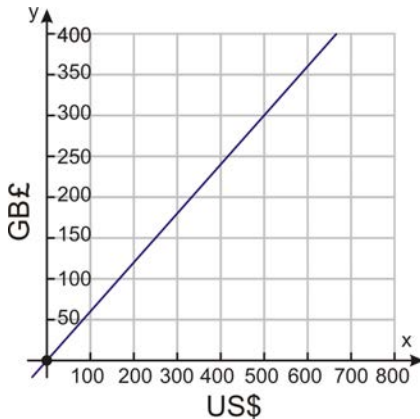
$$V = 37.5$$

At a depth of 20 inches, the volume of water in the tank is 37.5 gallons.

Example 3

The graph shown below is a conversion chart used to convert U.S. dollars (US\$) to British pounds (GB£) in a bank on a particular day. Use the chart to determine:

- a) the number of pounds you could buy for \$600
- b) the number of dollars it would cost to buy £200
- c) the exchange rate in pounds per dollar



Solution

We can read the answers to a) and b) right off the graph. It looks as if at $x = 600$ the graph is about one fifth of the way between £350 and £400. So \$600 would buy £360.

Similarly, the line $y = 200$ appears to intersect the graph about a third of the way between \$300 and \$400. We can round this to \$330, so it would cost approximately \$330 to buy £200.

To solve for the exchange rate, we should note that as this is a direct variation - the graph is a straight line passing through the origin. The slope of the line gives the constant of proportionality (in this case the **exchange rate**) and it is equal to the ratio of the y -value to x -value at any point. Looking closely at the graph, we can see that the line passes through one convenient lattice point: (500, 300). This will give us the most accurate value for the slope and so the exchange rate.

$$y = k \cdot x \Rightarrow \frac{y}{x} \quad \text{And so rate} = \frac{300 \text{ pounds}}{500 \text{ dollars}} = 0.60 \text{ pounds per dollar}$$

Graph Direct Variation Equations

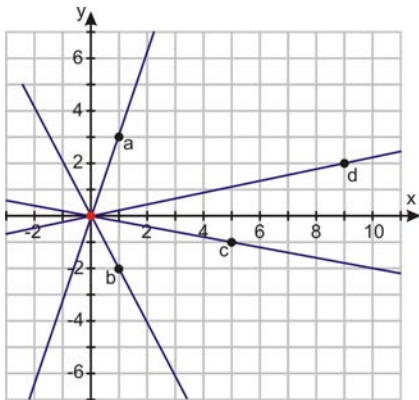
We know that all direct variation graphs pass through the origin, and also that the slope of the line is equal to the constant of proportionality, k . Graphing is a simple matter of using the point-slope or point-point methods discussed earlier in this chapter.

Example 4

Plot the following direct variations on the same graph.

- a) $y = 3x$
- b) $y = -2x$
- c) $y = -0.2x$
- d) $y = \frac{2}{9}x$

Solution



- a) The line passes through $(0, 0)$, as will all these functions. This function has a slope of 3. When we move across by one unit, the function increases by three units.
- b) The line has a slope of -2 . When we move across the graph by one unit, the function **falls** by two units.
- c) The line has a slope of -0.2 . As a fraction this is equal to $-\frac{1}{5}$. When we move across by five units, the function **falls** by one unit.
- d) The line passes through $(0, 0)$ and has a slope of $\frac{2}{9}$. When we move across the graph by 9 units, the function increases by two units.

For more examples of how to plot and identify direct variation functions, see the video at <http://neaportal.k12.ar.us/index.php/2010/06/slope-and-direct-variation/>.

Solve Real-World Problems Using Direct Variation Models

Direct variations are seen everywhere in everyday life. Any time one quantity increases at the same rate another quantity increases (for example, doubling when it doubles and tripling when it triples), we say that they follow a direct variation.

Newton's Second Law

In 1687 Sir Isaac Newton published the famous *Principia Mathematica*. It contained, among other things, his second law of motion. This law is often written as $F = m \cdot a$, where a force of F Newtons applied to a mass of m kilograms results in acceleration of a meters per second². Notice that if the mass stays constant, then this formula is basically the same as the direct variation equation, just with different variables—and m is the constant of proportionality.

Example 5

If a 175 Newton force causes a shopping cart to accelerate down the aisle with an acceleration of 2.5 m/s^2 , calculate:

- a) The mass of the shopping cart.
- b) The force needed to accelerate the same cart at 6 m/s^2 .

Solution

a) We can solve for m (the mass) by plugging in our given values for force and acceleration. $F = m \cdot a$ becomes $175 = m(2.5)$, and then we divide both sides by 2.5 to get $70 = m$.

So the mass of the shopping cart is 70 kg.

b) Once we have solved for the mass, we simply substitute that value, plus our required acceleration, back into the formula $F = m \cdot a$ and solve for F . We get $F = 70 \times 6 = 420$.

So the force needed to accelerate the cart at 6 m/s^2 is 420 Newtons.

Ohm's Law

The electrical current, I (amps), passing through an electronic component varies directly with the applied voltage, V (volts), according to the relationship $V = I \cdot R$, where R is the resistance (measured in Ohms). The resistance is considered to be a constant for all values of V and I , so once again, this formula is a version of the direct variation formula, with R as the constant of proportionality.

Example 6

A certain electronics component was found to pass a current of 1.3 amps at a voltage of 2.6 volts. When the voltage was increased to 12.0 volts the current was found to be 6.0 amps.

- Does the component obey Ohm's law?
- What would the current be at 6 volts?

Solution

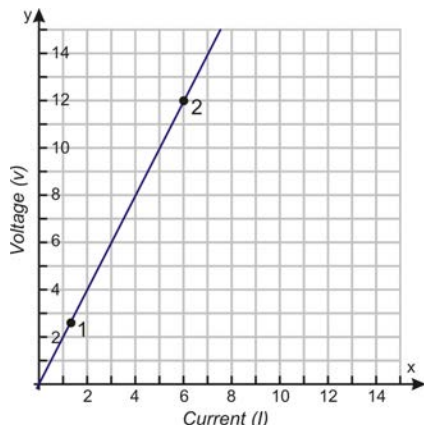
Ohm's law is a simple direct proportionality law, with the resistance R as our constant of proportionality. To know if this component obeys Ohm's law, we need to know if it follows a direct proportionality rule. In other words, is V directly proportional to I ?

We can determine this in two different ways.

Graph It: If we plot our two points on a graph and join them with a line, does the line pass through (0, 0)?

Voltage is the independent variable and current is the dependent variable, so normally we would graph V on the horizontal axis and I on the vertical axis. However, if we swap the variables around just this once, we'll get a graph whose slope conveniently happens to be equal to the resistance, R . So we'll treat I as the independent variable, and our two points will be (1.3, 2.6) and (6, 12).

Plotting those points and joining them gives the following graph:



The graph does appear to pass through the origin, so **yes, the component obeys Ohm's law.**

Solve for R: If this component does obey Ohm's law, the constant of proportionality (R) should be the same when we plug in the second set of values as when we plug in the first set. Let's see if it is. (We can quickly find the value of R in each case; since $V = I \cdot R$, that means $R = \frac{V}{I}$.)

$$\text{Case 1: } R = \frac{V}{I} = \frac{2.6}{1.3} = 2 \text{ Ohms}$$

$$\text{Case 2: } R = \frac{V}{I} = \frac{12}{6} = 2 \text{ Ohms}$$

The values for R agree! This means that we are indeed looking at a direct variation. **The component obeys Ohm's law.**

b) Now to find the current at 6 volts, simply substitute the values for V and R into $V = I \cdot R$. We found that $R = 2$, so we plug in 2 for R and 6 for V to get $6 = I(2)$, and divide both sides by 2 to get $3 = I$.

So **the current through the component at a voltage of 6 volts is 3 amps.**

Lesson Summary

- If a variable y varies *directly* with variable x , then we write the relationship as $y = k \cdot x$, where k is a constant called the **constant of proportionality**.
- **Direct variation** is very common in many areas of science.

Review Questions

- Plot the following direct variations on the same graph.
 - $y = \frac{4}{3}x$
 - $y = -\frac{2}{3}x$
 - $y = -\frac{1}{6}x$
 - $y = 1.75x$
- Dasan's mom takes him to the video arcade for his birthday.
 - In the first 10 minutes, he spends \$3.50 playing games. If his allowance for the day is \$20, how long can he keep playing games before his money is gone?
 - He spends the next 15 minutes playing Alien Invaders. In the first two minutes, he shoots 130 aliens. If he keeps going at this rate, how many aliens will he shoot in fifteen minutes?
 - The high score on this machine is 120000 points. If each alien is worth 100 points, will Dasan beat the high score? What if he keeps playing for five more minutes?
- The current standard for low-flow showerheads is 2.5 gallons per minute.
 - How long would it take to fill a 30-gallon bathtub using such a showerhead to supply the water?
 - If the bathtub drain were not plugged all the way, so that every minute 0.5 gallons ran out as 2.5 gallons ran in, how long would it take to fill the tub?
 - After the tub was full and the showerhead was turned off, how long would it take the tub to empty through the partly unplugged drain?
 - If the drain were immediately unplugged all the way when the showerhead was turned off, so that it drained at a rate of 1.5 gallons per minute, how long would it take to empty?
- Amin is using a hose to fill his new swimming pool for the first time. He starts the hose at 10 PM and leaves it running all night.
 - At 6 AM he measures the depth and calculates that the pool is four sevenths full. At what time will his new pool be full?
 - At 10 AM he measures again and realizes his earlier calculations were wrong. The pool is still only three quarters full. When will it actually be full?

- (c) After filling the pool, he needs to chlorinate it to a level of 2.0 ppm (parts per million). He adds two gallons of chlorine solution and finds that the chlorine level is now 0.7 ppm. How many more gallons does he need to add?
- (d) If the chlorine level in the pool decreases by 0.05 ppm per day, how much solution will he need to add each week?
5. Land in Wisconsin is for sale to property investors. A 232-acre lot is listed for sale for \$200,500.
- (a) Assuming the same price per acre, how much would a 60-acre lot sell for?
- (b) Again assuming the same price, what size lot could you purchase for \$100,000?
6. The force (F) needed to stretch a spring by a distance x is given by the equation $F = k \cdot x$, where k is the spring constant (measured in Newtons per centimeter, or N/cm). If a 12 Newton force stretches a certain spring by 10 cm, calculate:
- (a) The spring constant, k
- (b) The force needed to stretch the spring by 7 cm.
- (c) The distance the spring would stretch with a 23 Newton force.
7. Angela's cell phone is completely out of power when she puts it on the charger at 3 PM. An hour later, it is 30% charged. When will it be completely charged?
8. It costs \$100 to rent a recreation hall for three hours and \$150 to rent it for five hours.
- (a) Is this a direct variation?
- (b) Based on the cost to rent the hall for three hours, what would it cost to rent it for six hours, assuming it is a direct variation?
- (c) Based on the cost to rent the hall for five hours, what would it cost to rent it for six hours, assuming it is a direct variation?
- (d) Plot the costs given for three and five hours and graph the line through those points. Based on that graph, what would you expect the cost to be for a six-hour rental?

4.7 Linear Function Graphs

Learning Objectives

- Recognize and use function notation.
- Graph a linear function.
- Analyze arithmetic progressions.

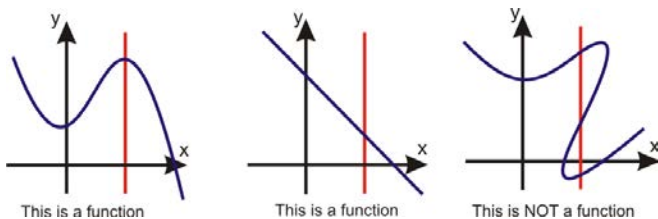
Introduction

The highly exclusive Fellowship of the Green Mantle allows in only a limited number of new members a year. In its third year of membership it has 28 members, in its fourth year it has 33, and in its fifth year it has 38. How many members are admitted a year, and how many founding members were there?

Functions

So far we've used the term **function** to describe many of the equations we've been graphing, but in mathematics it's important to remember that not all equations are functions. In order to be a function, a relationship between two variables, x and y , must map each x -value to **exactly one** y -value.

Visually this means the graph of y versus x must pass the **vertical line test**, meaning that a vertical line drawn through the graph of the function must never intersect the graph in more than one place:



Use Function Notation

When we write functions we often use the notation “ $f(x) =$ ” in place of “ $y =$ ”. $f(x)$ is pronounced “ f of x ”.

Example 1

Rewrite the following equations so that y is a function of x and is written $f(x)$:

- $y = 2x + 5$
- $y = -0.2x + 7$
- $x = 4y - 5$
- $9x + 3y = 6$

Solution

- Simply replace y with $f(x)$: $f(x) = 2x + 5$
- Again, replace y with $f(x)$: $f(x) = -0.2x + 7$
- First we need to solve for y . Starting with $x = 4y - 5$, we add 5 to both sides to get $x + 5 = 4y$, divide by 4 to get $\frac{x+5}{4} = y$, and then replace y with $f(x)$: $f(x) = \frac{x+5}{4}$.
- Solve for y : take $9x + 3y = 6$, subtract $9x$ from both sides to get $3y = 6 - 9x$, divide by 3 to get $y = \frac{6-9x}{3} = 2 - 3x$, and express as a function: $f(x) = 2 - 3x$.

Using the functional notation in an equation gives us more information. For instance, the expression $f(x) = mx + b$ shows clearly that x is the independent variable because you **plug in** values of x into the function and perform a series of operations on the value of x in order to calculate the values of the dependent variable, y .

We can also plug in expressions rather than just numbers. For example, if our function is $f(x) = x + 2$, we can plug in the expression $(x + 5)$. We would express this as $f(x + 5) = (x + 5) + 2 = x + 7$.

Example 2

A function is defined as $f(x) = 6x - 36$. Evaluate the following:

- $f(2)$
- $f(0)$
- $f(z)$
- $f(x + 3)$
- $f(2r - 1)$

Solution

- Substitute $x = 2$ into the function $f(x)$: $f(2) = 6 \cdot 2 - 36 = 12 - 36 = -24$
- Substitute $x = 0$ into the function $f(x)$: $f(0) = 6 \cdot 0 - 36 = 0 - 36 = -36$

c) Substitute $x = z$ into the function $f(x) : f(z) = 6z + 36$

d) Substitute $x = (x + 3)$ into the function $f(x) : f(x + 3) = 6(x + 3) + 36 = 6x + 18 + 36 = 6x + 54$

e) Substitute $x = (2r + 1)$ into the function $f(x) : f(2r + 1) = 6(2r + 1) + 36 = 12r + 6 + 36 = 12r + 42$

Graph a Linear Function

Since the notations “ $f(x) =$ ” and “ $y =$ ” are interchangeable, we can use all the concepts we have learned so far to graph functions.

Example 3

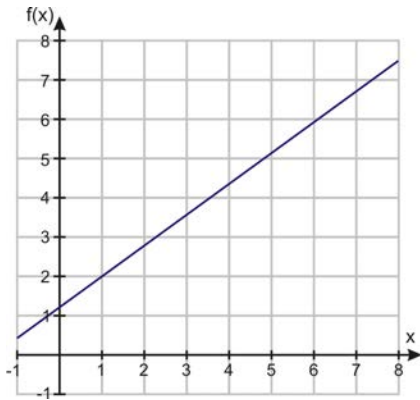
Graph the function $f(x) = \frac{3x+5}{4}$.

Solution

We can write this function in **slope-intercept** form:

$$f(x) = \frac{3}{4}x + \frac{5}{4} = 0.75x + 1.25$$

So our graph will have a y -intercept of $(0, 1.25)$ and a slope of 0.75.



Example 4

Graph the function $f(x) = \frac{7(5-x)}{5}$.

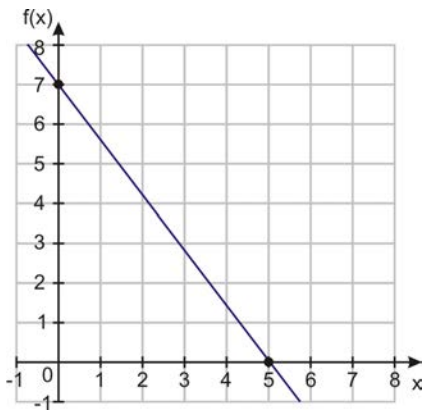
Solution

This time we'll solve for the x - and y -intercepts.

To solve for the y -intercept, plug in $x = 0 : f(0) = \frac{7(5-0)}{5} = \frac{35}{5} = 7$, so the y -intercept is **(0, 7)**.

To solve for the x -intercept, set $f(x) = 0 : 0 = \frac{7(5-x)}{5}$, so $0 = 35 - 7x$, therefore $7x = 35$ and $x = 5$. The x -intercept is **(5, 0)**.

We can graph the function from those two points:



Arithmetic Progressions

You may have noticed that with linear functions, when you increase the x -value by 1 unit, the y -value increases by a fixed amount, equal to the slope. For example, if we were to make a table of values for the function $f(x) = 2x + 3$, we might start at $x = 0$ and then add 1 to x for each row:

Table 4.6:

x	$f(x)$
0	3
1	5
2	7
3	9
4	11

Notice that the values for $f(x)$ go up by 2 (the slope) each time. When we repeatedly add a fixed value to a starting number, we get a sequence like $\{3, 5, 7, 9, 11, \dots\}$. We call this an **arithmetic progression**, and it is characterized by the fact that each number is bigger (or smaller) than the preceding number by a fixed amount. This amount is called the **common difference**. We can find the common difference for a given sequence by taking 2 consecutive terms in the sequence and subtracting the first from the second.

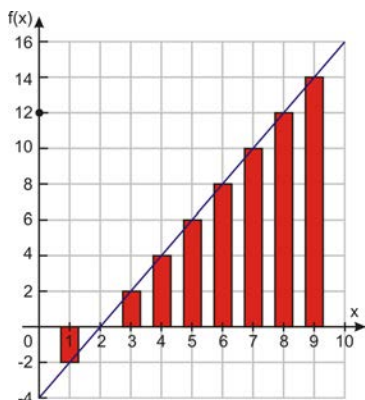
Example 5

Find the common difference for the following arithmetic progressions:

- $\{7, 11, 15, 19, \dots\}$
- $\{12, 1, -10, -21, \dots\}$
- $\{7, _, 12, _, 17, \dots\}$

Solution

- $11 - 7 = 4$; $15 - 11 = 4$; $19 - 15 = 4$. The common difference is **4**.
- $1 - 12 = -11$. The common difference is **-11**.
- There are not 2 consecutive terms here, but we know that to get the term after 7 we would add the common difference, and then to get to 12 we would add the common difference again. So *twice* the common difference is $12 - 7 = 5$, and so the common difference is **2.5**.



Arithmetic sequences and linear functions are very closely related. To get to the next term in a arithmetic sequence, you add the *common difference* to the last term; similarly, when the x -value of a linear function increases by one, the y -value increases by the amount of the *slope*. So arithmetic sequences are very much like linear functions, with the common difference playing the same role as the slope.

The graph below shows the arithmetic progression $\{-2, 0, 2, 4, 6\dots\}$ along with the function $y = 2x - 4$. The only major difference between the two graphs is that an arithmetic sequence is **discrete** while a linear function is **continuous**.

We can write a formula for an arithmetic progression: if we define the first term as a_1 and d as the common difference, then the other terms are as follows:

$$\begin{array}{cccccc}
 a_1 & a_2 & a_3 & a_4 & a_5 & a_n \\
 a_1 & a_1 + d & a_1 + 2d & a_1 + 3d & a_1 + 4d & \dots & a_1 + (n - 1) \cdot d
 \end{array}$$

The online calculator at <http://planetcalc.com/177/> will tell you the n th term in an arithmetic progression if you tell it the first term, the common difference, and what value to use for n (in other words, which term in the sequence you want to know). It will also tell you the sum of all the terms up to that point. Finding sums of sequences is something you will learn to do in future math classes.

Lesson Summary

- In order for an equation to be a **function**, the relationship between the two variables, x and y , must map each x -value to *exactly* one y -value.
- The graph of a function of y versus x must pass the **vertical line test**: any vertical line will only cross the graph of the function in one place.
- Functions can be expressed in function notation using $f(x) =$ in place of $y =$.
- The sequence of $f(x)$ values for a linear function form an arithmetic progression. Each number is greater than (or less than) the preceding number by a fixed amount, or **common difference**.

Review Questions

1. When an object falls under gravity, it gains speed at a constant rate of 9.8 m/s every second. An item dropped from the top of the Eiffel Tower, which is 300 meters tall, takes 7.8 seconds to hit the ground. How fast is it moving on impact?
2. A prepaid phone card comes with \$20 worth of calls on it. Calls cost a flat rate of \$0.16 per minute.
 - (a) Write the value left on the card as a function of minutes used so far.
 - (b) Use the function to determine how many minutes of calls you can make with the card.

3. For each of the following functions evaluate:

(a) $f(x) = -2x + 3$

(b) $f(x) = 0.7x + 3.2$

(c) $f(x) = \frac{5(2-x)}{11}$

i. $f(-3)$

ii. $f(0)$

iii. $f(z)$

iv. $f(x + 3)$

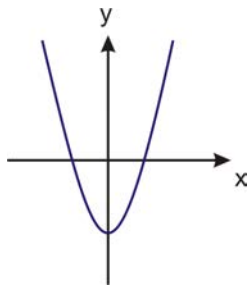
v. $f(2n)$

vi. $f(3y + 8)$

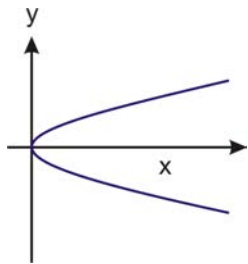
vii. $f\left(\frac{9}{2}\right)$

4. Determine whether the following could be graphs of **functions**.

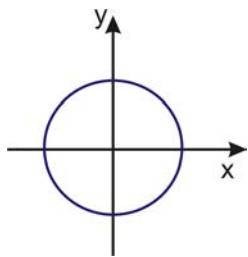
(a)



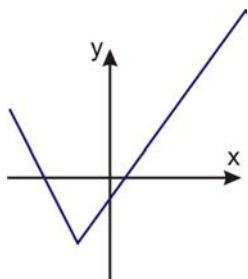
(b)



(c)



(d)



5. The roasting guide for a turkey suggests cooking for 100 minutes plus an additional 8 minutes per pound.

(a) Write a function for the roasting time the given the turkey weight in pounds (x).

(b) Determine the time needed to roast a 10 lb turkey.

- (c) Determine the time needed to roast a 27 lb turkey.
 - (d) Determine the maximum size turkey you could roast in 4.5 hours.
6. Determine the missing terms in the following arithmetic progressions.
- (a) $\{-11, 17, _, 73\}$
 - (b) $\{2, _, -4\}$
 - (c) $\{13, _, _, _, 0\}$

4.8 Problem-Solving Strategies - Graphs

Learning Objectives

- Read and understand given problem situations.
- Use the strategy “Read a Graph.”
- Use the strategy “Make a Graph.”
- Solve real-world problems using selected strategies as part of a plan.

Introduction

In this chapter, we’ve been solving problems where quantities are linearly related to each other. In this section, we’ll look at a few examples of linear relationships that occur in real-world problems, and see how we can solve them using graphs. Remember back to our Problem Solving Plan:

1. **Understand the Problem**
2. **Devise a Plan—Translate**
3. **Carry Out the Plan—Solve**
4. **Look—Check and Interpret**

Example 1

A cell phone company is offering its costumers the following deal: You can buy a new cell phone for \$60 and pay a monthly flat rate of \$40 per month for unlimited calls. How much money will this deal cost you after 9 months?

Solution

Let’s follow the problem solving plan.

Step 1: The phone costs \$60; the calling plan costs \$40 per month.

Let x = number of months.

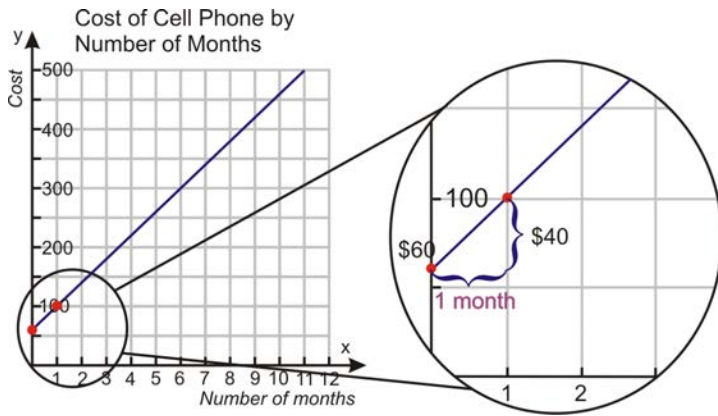
Let y = total cost.

Step 2: We can solve this problem by making a graph that shows the number of months on the horizontal axis and the cost on the vertical axis.

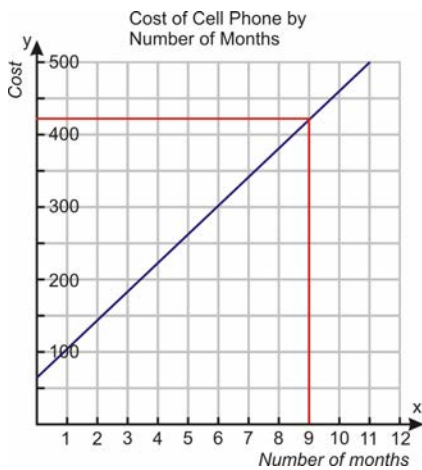
Since you pay \$60 when you get the phone, the y -intercept is $(0, 60)$.

You pay \$40 for each month, so the cost rises by \$40 for 1 month, so the slope is 40.

We can graph this line using the slope-intercept method.



Step 3: The question was “How much will this deal cost after 9 months?” We can now read the answer from the graph. We draw a vertical line from 9 months until it meets the graph, and then draw a horizontal line until it meets the vertical axis.



We see that after 9 months **you pay approximately \$420.**

Step 4: To check if this is correct, let’s think of the deal again.

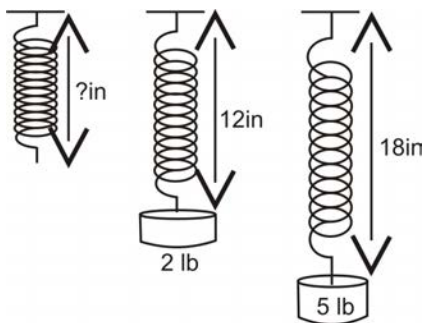
Originally, you pay \$60 and then \$40 a month for 9 months.

$$\begin{aligned} \text{Phone} &= \$60 \\ \text{Calling plan} &= \$40 \times 9 = \$360 \\ \text{Total cost} &= \$420. \end{aligned}$$

The answer checks out.

Example 2

A stretched spring has a length of 12 inches when a weight of 2 lbs is attached to the spring. The same spring has a length of 18 inches when a weight of 5 lbs is attached to the spring. What is the length of the spring when no weights are attached?



Solution

Step 1: We know: the length of the spring = 12 inches when weight = 2 lbs

the length of the spring = 18 inches when weight = 5 lbs

We want: the length of the spring when weight = 0 lbs

Let x = the weight attached to the spring.

Let y = the length of the spring.

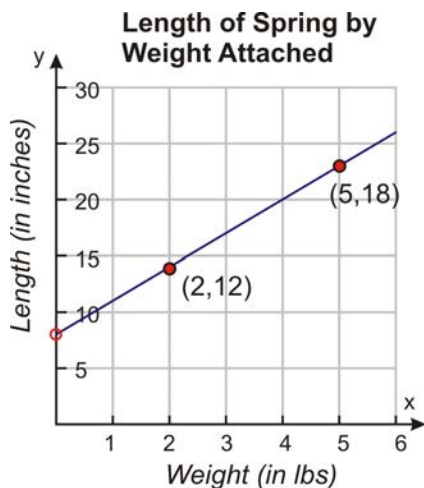
Step 2: We can solve this problem by making a graph that shows the weight on the horizontal axis and the length of the spring on the vertical axis.

We have two points we can graph:

When the weight is 2 lbs, the length of the spring is 12 inches. This gives point (2, 12).

When the weight is 5 lbs, the length of the spring is 18 inches. This gives point (5, 18).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: “What is the length of the spring when no weights are attached?”

We can answer this question by reading the graph we just made. When there is no weight on the spring, the x -value equals zero, so we are just looking for the y -intercept of the graph. On the graph, the y -intercept appears to be approximately 8 inches.

Step 4: To check if this correct, let’s think of the problem again.

You can see that the length of the spring goes up by 6 inches when the weight is increased by 3 lbs, so the slope of the line is $\frac{6 \text{ inches}}{3 \text{ lbs}} = 2 \text{ inches/lb}$.

To find the length of the spring when there is no weight attached, we can look at the spring when there are 2 lbs attached. For each pound we take off, the spring will shorten by 2 inches. If we take off 2 lbs, the spring will be shorter by 4 inches. So, the length of the spring with no weights is 12 inches - 4 inches = 8 inches.

The answer checks out.

Example 3

Christine took 1 hour to read 22 pages of Harry Potter. She has 100 pages left to read in order to finish the book. How much time should she expect to spend reading in order to finish the book?

Solution

Step 1: We know - Christine takes 1 hour to read 22 pages.

We want - How much time it takes to read 100 pages.

Let x = the time expressed in hours.

Let y = the number of pages.

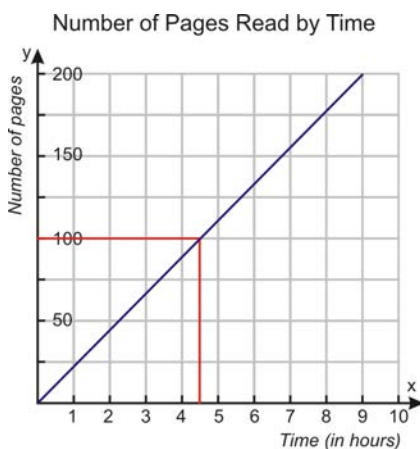
Step 2: We can solve this problem by making a graph that shows the number of hours spent reading on the horizontal axis and the number of pages on the vertical axis.

We have two points we can graph:

Christine takes 1 hour to read 22 pages. This gives point (1, 22).

A second point is not given, but we know that Christine would take 0 hours to read 0 pages. This gives point (0, 0).

Graphing those two points and connecting them gives us our line.



Step 3: The question was: “How much time should Christine expect to spend reading 100 pages?” We can find the answer from reading the graph - we draw a horizontal line from 100 pages until it meets the graph and then we draw the vertical until it meets the horizontal axis. We see that it takes **approximately 4.5 hours** to read the remaining 100 pages.

Step 4: To check if this correct, let’s think of the problem again.

We know that Christine reads 22 pages per hour - this is the slope of the line or the rate at which she is reading. To find how many hours it takes her to read 100 pages, we divide the number of pages by the rate. In this case, $\frac{100 \text{ pages}}{22 \text{ pages/hour}} = 4.54 \text{ hours}$. This is very close to the answer we got from reading the graph.

The answer checks out.

Example 4

Aatif wants to buy a surfboard that costs \$249. He was given a birthday present of \$50 and he has a summer job that pays him \$6.50 per hour. To be able to buy the surfboard, how many hours does he need to work?

Solution

Step 1: We know - The surfboard costs \$249.

Aatif has \$50.

His job pays \$6.50 per hour.

We want - How many hours Aatif needs to work to buy the surfboard.

Let x = the time expressed in hours

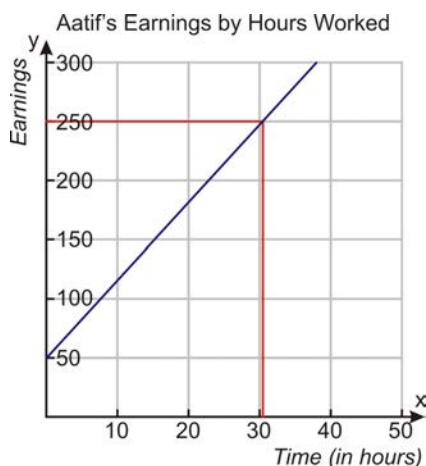
Let y = Aatif's earnings

Step 2: We can solve this problem by making a graph that shows the number of hours spent working on the horizontal axis and Aatif's earnings on the vertical axis.

Aatif has \$50 at the beginning. This is the y -intercept: $(0, 50)$.

He earns \$6.50 per hour. This is the slope of the line.

We can graph this line using the slope-intercept method. We graph the y -intercept of $(0, 50)$, and we know that for each unit in the horizontal direction, the line rises by 6.5 units in the vertical direction. Here is the line that describes this situation.



Step 3: The question was: “How many hours does Aatif need to work to buy the surfboard?”

We find the answer from reading the graph - since the surfboard costs \$249, we draw a horizontal line from \$249 on the vertical axis until it meets the graph and then we draw a vertical line downwards until it meets the horizontal axis. We see that it takes **approximately 31 hours** to earn the money.

Step 4: To check if this correct, let's think of the problem again.

We know that Aatif has \$50 and needs \$249 to buy the surfboard. So, he needs to earn $\$249 - \$50 = \$199$ from his job.

His job pays \$6.50 per hour. To find how many hours he need to work we divide: $\frac{\$199}{\$6.50/\text{hour}} = 30.6 \text{ hours}$. This is very close to the answer we got from reading the graph.

The answer checks out.

Lesson Summary

The four steps of the **problem solving plan** when using graphs are:

1. **Understand the Problem**
2. **Devise a Plan—Translate:** Make a graph.
3. **Carry Out the Plan—Solve:** Use the graph to answer the question asked.
4. **Look—Check and Interpret**

Review Questions

Solve the following problems by making a graph and reading it.

1. A gym is offering a deal to new members. Customers can sign up by paying a registration fee of \$200 and a monthly fee of \$39.
 - (a) How much will this membership cost a member by the end of the year?
 - (b) The old membership rate was \$49 a month with a registration fee of \$100. How much more would a year's membership cost at that rate?
 - (c) **Bonus:** For what number of months would the two membership rates be the same?
2. A candle is burning at a linear rate. The candle measures five inches two minutes after it was lit. It measures three inches eight minutes after it was lit.
 - (a) What was the original length of the candle?
 - (b) How long will it take to burn down to a half-inch stub?
 - (c) Six half-inch stubs of candle can be melted together to make a new candle measuring $2\frac{5}{6}$ inches (a little wax gets lost in the process). How many stubs would it take to make three candles the size of the original candle?
3. A dipped candle is made by taking a wick and dipping it repeatedly in melted wax. The candle gets a little bit thicker with each added layer of wax. After it has been dipped three times, the candle is 6.5 mm thick. After it has been dipped six times, it is 11 mm thick.
 - (a) How thick is the wick before the wax is added?
 - (b) How many times does the wick need to be dipped to create a candle 2 cm thick?
4. Tali is trying to find the thickness of a page of his telephone book. In order to do this, he takes a measurement and finds out that 55 pages measures $\frac{1}{8}$ inch. What is the thickness of one page of the phone book?
5. Bobby and Petra are running a lemonade stand and they charge 45 cents for each glass of lemonade. In order to break even they must make \$25.
 - (a) How many glasses of lemonade must they sell to break even?
 - (b) When they've sold \$18 worth of lemonade, they realize that they only have enough lemons left to make 10 more glasses. To break even now, they'll need to sell those last 10 glasses at a higher price. What does the new price need to be?
6. Dale is making cookies using a recipe that calls for 2.5 cups of flour for two dozen cookies. How many cups of flour does he need to make five dozen cookies?
7. To buy a car, Jason makes a down payment of \$1500 and pays \$350 per month in installments.
 - (a) How much money has Jason paid at the end of one year?
 - (b) If the total cost of the car is \$8500, how long will it take Jason to finish paying it off?
 - (c) The resale value of the car decreases by \$100 each month from the original purchase price. If Jason sells the car as soon as he finishes paying it off, how much will he get for it?

8. Anne transplants a rose seedling in her garden. She wants to track the growth of the rose so she measures its height every week. On the third week, she finds that the rose is 10 inches tall and on the eleventh week she finds that the rose is 14 inches tall. Assuming the rose grows linearly with time, what was the height of the rose when Anne planted it?
9. Ravi hangs from a giant spring whose length is 5 m. When his child Nimi hangs from the spring its length is 2 m. Ravi weighs 160 lbs and Nimi weighs 40 lbs. Write the equation for this problem in slope-intercept form. What should we expect the length of the spring to be when his wife Amardeep, who weighs 140 lbs, hangs from it?
10. Nadia is placing different weights on a spring and measuring the length of the stretched spring. She finds that for a 100 gram weight the length of the stretched spring is 20 cm and for a 300 gram weight the length of the stretched spring is 25 cm.
 - (a) What is the unstretched length of the spring?
 - (b) If the spring can only stretch to twice its unstretched length before it breaks, how much weight can it hold?
11. Andrew is a submarine commander. He decides to surface his submarine to periscope depth. It takes him 20 minutes to get from a depth of 400 feet to a depth of 50 feet.
 - (a) What was the submarine's depth five minutes after it started surfacing?
 - (b) How much longer would it take at that rate to get all the way to the surface?
12. Kiersta's phone has completely run out of battery power when she puts it on the charger. Ten minutes later, when the phone is 40% recharged, Kiersta's friend Danielle calls and Kiersta takes the phone off the charger to talk to her. When she hangs up 45 minutes later, her phone has 10% of its charge left. Then she gets another call from her friend Kwan.
 - (a) How long can she spend talking to Kwan before the battery runs out again?
 - (b) If she puts the phone back on the charger afterward, how long will it take to recharge completely?
13. Marji is painting a 75-foot fence. She starts applying the first coat of paint at 2 PM, and by 2:10 she has painted 30 feet of the fence. At 2:15, her husband, who paints about $\frac{2}{3}$ as fast as she does, comes to join her.
 - (a) How much of the fence has Marji painted when her husband joins in?
 - (b) When will they have painted the whole fence?
 - (c) How long will it take them to apply the second coat of paint if they work together the whole time?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9614>.

Chapter 5

Writing Linear Equations

5.1 Forms of Linear Equations

Learning Objectives

- Write equations in slope-intercept form.
- Write equations in point-slope form.
- Write equations in standard form.
- Solve real-world problems using linear models in all three forms.

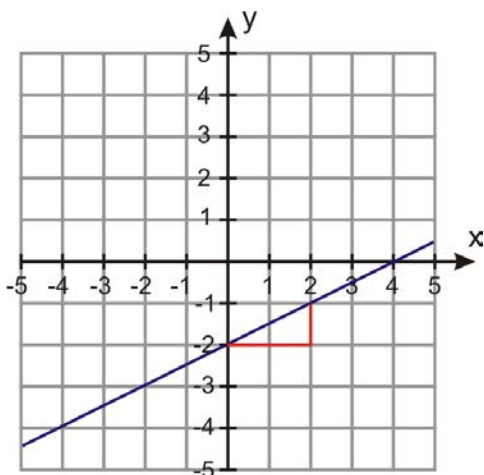
Introduction

We saw in the last chapter that many real-world situations can be described with linear graphs and equations. In this chapter, we'll see how to find those equations in a variety of situations.

Write an Equation Given Slope and y -Intercept

You've already learned how to write an equation in slope-intercept form: simply start with the general equation for the slope-intercept form of a line, $y = mx + b$, and then plug the given values of m and b into the equation. For example, a line with a slope of 4 and a y -intercept of -3 would have the equation $y = 4x - 3$.

If you are given just the graph of a line, you can read off the slope and y -intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is $\frac{1}{2}$. Also, you can see that the y -intercept is -2, so the equation of the line is $y = \frac{1}{2}x - 2$.



Write an Equation Given the Slope and a Point

Often, we don't know the value of the y -intercept, but we know the value of y for a non-zero value of x . In this case, it's often easier to write an equation of the line in **point-slope form**. An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where m is the slope and (x_0, y_0) is a point on the line.

Example 1

A line has a slope of $\frac{3}{5}$, and the point $(2, 6)$ is on the line. Write the equation of the line in point-slope form.

Solution

Start with the formula $y - y_0 = m(x - x_0)$.

Plug in $\frac{3}{5}$ for m , 2 for x_0 and 6 for y_0 .

The equation in point-slope form is $y - 6 = \frac{3}{5}(x - 2)$.

Notice that the equation in point-slope form is not solved for y . If we did solve it for y , we'd have it in y -intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

Write an Equation Given Two Points

Point-slope form also comes in useful when we need to find an equation given just two points on a line.

For example, suppose we are told that the line passes through the points $(-2, 3)$ and $(5, 2)$. To find the equation of the line, we can start by finding the slope.

Starting with the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, we plug in the x - and y -values of the two points to get $m = \frac{2 - 3}{5 - (-2)} = \frac{-1}{7}$. We can plug that value of m into the point-slope formula to get $y - y_0 = -\frac{1}{7}(x - x_0)$.

Now we just need to pick one of the two points to plug into the formula. Let's use $(5, 2)$; that gives us $y - 2 = -\frac{1}{7}(x - 5)$.

What if we'd picked the other point instead? Then we'd have ended up with the equation $y - 3 = -\frac{1}{7}(x + 2)$, which doesn't look the same. That's because there's more than one way to write an equation for a given line in point-slope form. But let's see what happens if we solve each of those equations for y .

Starting with $y - 2 = -\frac{1}{7}(x - 5)$, we distribute the $-\frac{1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$,

$$\text{or } y = -\frac{1}{7}x + \frac{19}{7}.$$

On the other hand, if we start with $y - 3 = -\frac{1}{7}(x + 2)$, we need to distribute the $-\frac{1}{7}$ and add 3 to both sides. That gives us $y = -\frac{1}{7}x - \frac{2}{7} + 3$, which also simplifies to $y = -\frac{1}{7}x + \frac{19}{7}$.

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to y-intercept form.

Example 2

A line contains the points $(3, 2)$ and $(-2, 4)$. Write an equation for the line in point-slope form; then write an equation in y-intercept form.

Solution

Find the slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{-2 - 3} = -\frac{2}{5}$

Plug in the value of the slope: $y - y_0 = -\frac{2}{5}(x - x_0)$.

Plug point $(3, 2)$ into the equation: $y - 2 = -\frac{2}{5}(x - 3)$.

The equation in point-slope form is $y - 2 = -\frac{2}{5}(x - 3)$.

To convert to y-intercept form, simply solve for y:

$$y - 2 = -\frac{2}{5}(x - 3) \rightarrow y - 2 = -\frac{2}{5}x - \frac{6}{5} \rightarrow y = -\frac{2}{5}x - \frac{6}{5} + 2 = -\frac{2}{5}x + \frac{4}{5}.$$

The equation in y-intercept form is $y = -\frac{2}{5}x + \frac{4}{5}$.

Graph an Equation in Point-Slope Form

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation $y - y_0 = m(x - x_0)$, you can just read off the slope m and the point (x_0, y_0) . To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

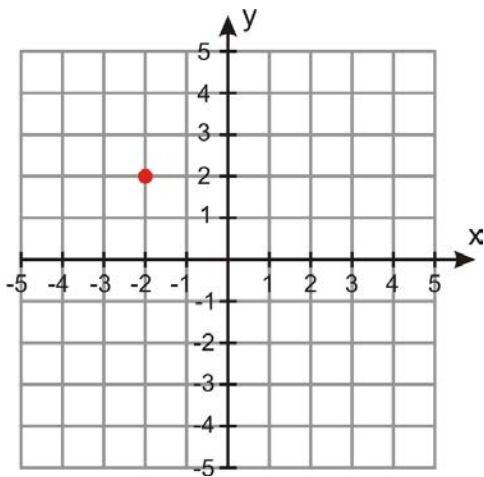
Example 5

Make a graph of the line given by the equation $y + 2 = \frac{2}{3}(x - 2)$.

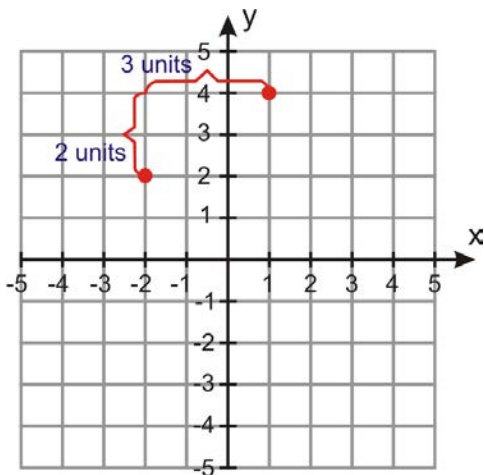
Solution

To read off the right values, we need to rewrite the equation slightly: $y - (-2) = \frac{2}{3}(x - 2)$. Now we see that point $(2, -2)$ is on the line and that the slope is $\frac{2}{3}$.

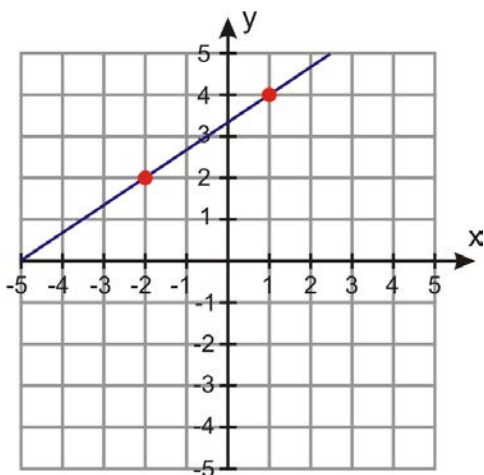
First plot point $(2, -2)$ on the graph:



A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:



Now draw a line through the two points and extend it in both directions:



Linear Equations in Standard Form

You've already encountered another useful form for writing linear equations: **standard form**. An equation in standard form is written $ax + by = c$, where a, b , and c are all integers and a is positive. (Note that the b in the standard form is different than the b in the slope-intercept form.)

One useful thing about standard form is that it allows us to write equations for vertical lines, which we can't do in slope-intercept form.

For example, let's look at the line that passes through points $(2, 6)$ and $(2, 9)$. How would we find an equation for that line in slope-intercept form?

First we'd need to find the slope: $m = \frac{9-6}{0-0} = \frac{3}{0}$. But that slope is undefined because we can't divide by zero. And if we can't find the slope, we can't use point-slope form either.

If we just graph the line, we can see that x equals 2 no matter what y is. There's no way to express that in slope-intercept or point-slope form, but in standard form we can just say that $x + 0y = 2$, or simply $x = 2$.

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of x is not negative.

Example 1

Rewrite the following equations in standard form:

a) $y = 5x - 7$

b) $y - 2 = -3(x + 3)$

c) $y = \frac{2}{3}x + \frac{1}{2}$

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of x is not negative.

a) $y = 5x - 7$

Subtract y from both sides to get $0 = 5x - y - 7$.

Add 7 to both sides to get $7 = 5x - y$.

Flip the equation around to put it in standard form: $5x - y = 7$.

b) $y - 2 = -3(x + 3)$

Distribute the -3 on the right-hand-side to get $y - 2 = -3x - 9$.

Add $3x$ to both sides to get $y + 3x - 2 = -9$.

Add 2 to both sides to get $y + 3x = -7$. Flip that around to get $3x + y = -7$.

c) $y = \frac{2}{3}x + \frac{1}{2}$

Find the common denominator for all terms in the equation – in this case that would be 6.

Multiply all terms in the equation by 6: $6\left(y = \frac{2}{3}x + \frac{1}{2}\right) \Rightarrow 6y = 4x + 3$

Subtract $6y$ from both sides: $0 = 4x - 6y + 3$

Subtract 3 from both sides: $-3 = 4x - 6y$

The equation in standard form is $4x - 6y = -3$.

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there's an even easier way. Let's look at what happens when we rewrite an equation in standard form.

Starting with the equation $ax + by = c$, we would subtract ax from both sides to get $by = -ax + c$. Then we would divide all terms by b and end up with $y = -\frac{a}{b}x + \frac{c}{b}$.

That means that the slope is $-\frac{a}{b}$ and the y -intercept is $\frac{c}{b}$. So next time we look at an equation in standard form, we don't have to rewrite it to find the slope; we know the slope is just $-\frac{a}{b}$, where a and b are the coefficients of x and y in the equation.

Example 2

Find the slope and the y -intercept of the following equations written in standard form.

a) $3x + 5y = 6$

b) $2x - 3y = -8$

c) $x - 5y = 10$

Solution

a) $a = 3, b = 5$, and $c = 6$, so the slope is $-\frac{a}{b} = -\frac{3}{5}$, and the y -intercept is $\frac{c}{b} = \frac{6}{5}$.

b) $a = 2, b = -3$, and $c = -8$, so the slope is $-\frac{a}{b} = \frac{2}{3}$, and the y -intercept is $\frac{c}{b} = \frac{8}{3}$.

c) $a = 1, b = -5$, and $c = 10$, so the slope is $-\frac{a}{b} = \frac{1}{5}$, and the y -intercept is $\frac{c}{b} = \frac{10}{-5} = -2$.

Once we've found the slope and y -intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let's graph the line given by the equation $3x - 2y = 6$.

To find the x -intercept, cover up the y term (remember, the x -intercept is where $y = 0$):

$$3x - \text{[cover up]} = 6$$

$$3x = 6 \Rightarrow x = 2$$

The x -intercept is $(2, 0)$.

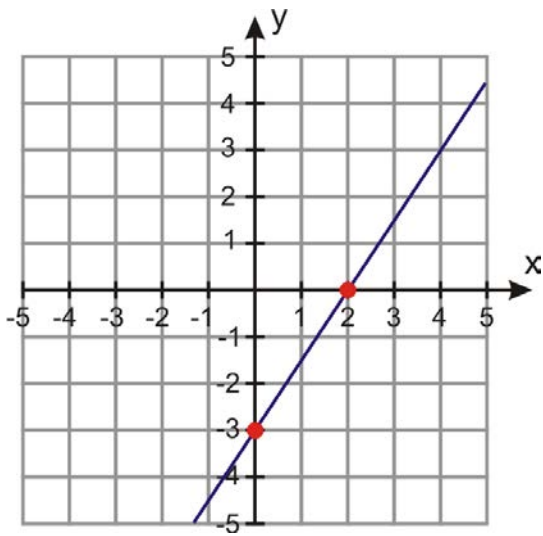
To find the y -intercept, cover up the x term (remember, the y -intercept is where $x = 0$):

$$\text{[cover up]} - 2y = 6$$

$$-2y = 6 \Rightarrow y = -3$$

The y -intercept is $(0, -3)$.

We plot the intercepts and draw a line through them that extends in both directions:

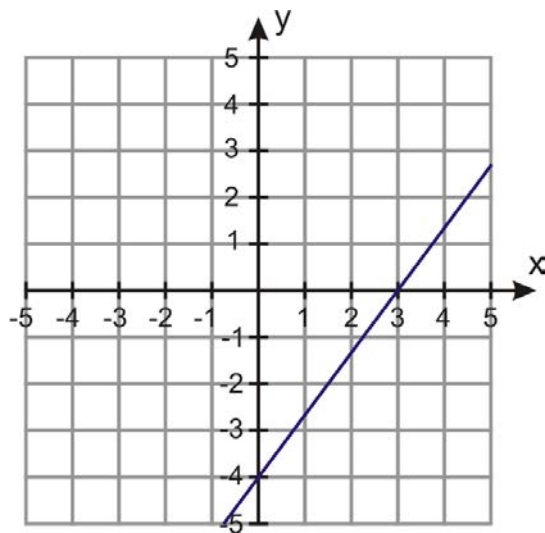


Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.

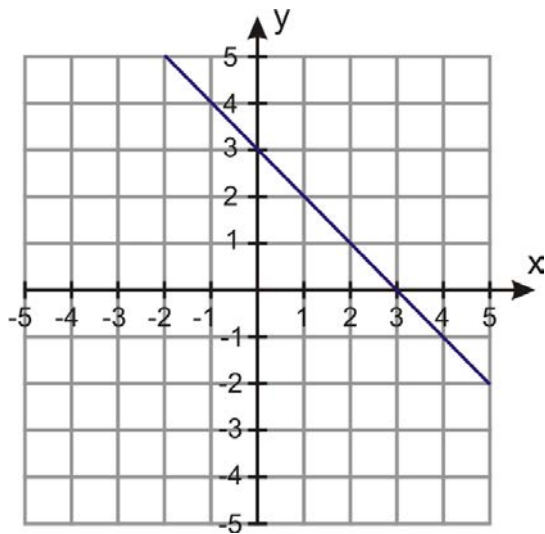
Example 3

Find the equation of each line and write it in standard form.

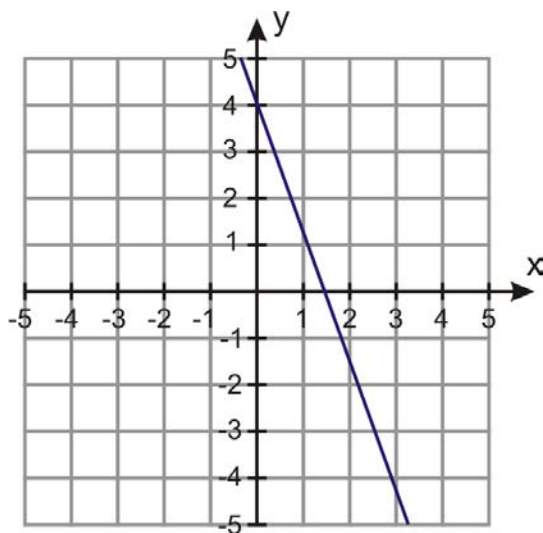
a)



b)



c)



Solution

a) We see that the x -intercept is $(3, 0) \Rightarrow x = 3$ and the y -intercept is $(0, -4) \Rightarrow y = -4$

We saw that in standard form $ax + by = c$: if we “cover up” the y term, we get $ax = c$, and if we “cover up” the x term, we get $by = c$.

So we need to find values for a and b so that we can plug in 3 for x and -4 for y and get the same value for c in both cases. This is like finding the least common multiple of the x - and y -intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by -3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and **the equation in standard form is $4x - 3y = 12$.**

b) We see that the x -intercept is $(3, 0) \Rightarrow x = 3$ and the y -intercept is $(0, 3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. **The equation in standard form is $x + y = 3$.**

c) We see that the x -intercept is $(\frac{3}{2}, 0) \Rightarrow x = \frac{3}{2}$ and the y -intercept is $(0, 4) \Rightarrow y = 4$

Let's multiply the x -intercept equation by 2 $\Rightarrow 2x = 3$

Then we see we can multiply the x -intercept again by 4 and the y -intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.

Solving Real-World Problems Using Linear Models in Point-Slope Form

Let's solve some word problems where we need to write the equation of a straight line in point-slope form.

Example 4

Marciel rented a moving truck for the day. Marciel only remembers that the rental truck company charges \$40 per day and some number of cents per mile. Marciel drives 46 miles and the final amount of the bill (before tax) is \$63. What is the amount per mile the truck rental company charges? Write an equation in point-slope form that describes this situation. How much would it cost to rent this truck if Marciel drove 220 miles?

Solution

Let's define our variables:

x = distance in miles

y = cost of the rental truck

Peter pays a flat fee of \$40 for the day; this is the y -intercept.

He pays \$63 for 46 miles; this is the coordinate point (46,63).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the coordinate point: $63 - y_0 = m(46 - x_0)$

Plug in the point (0, 40): $63 - 40 = m(46 - 0)$

Solve for the slope: $23 = 46m \rightarrow m = \frac{23}{46} = 0.5$

The slope is 0.5 dollars per mile, so the truck company charges 50 cents per mile ($\$0.5 = 50$ cents). Plugging in the slope and the y -intercept, the equation of the line is $y = 0.5x + 40$.

To find out the cost of driving the truck 220 miles, we plug in $x = 220$ to get $y - 40 = 0.5(220) \Rightarrow y = \150 .

Driving 220 miles would cost \$150.

Example 5

Anne got a job selling window shades. She receives a monthly base salary and a \$6 commission for each window shade she sells. At the end of the month she adds up sales and she figures out that she sold 200 window shades and made \$2500. Write an equation in point-slope form that describes this situation. How much is Anne's monthly base salary?

Solution

Let's define our variables:

x = number of window shades sold

y = Anne's earnings

We see that we are given the slope and a point on the line:

Nadia gets \$6 for each shade, so the slope is 6.

She made \$2500 when she sold 200 shades, so the point is (200, 2500).

Start with the point-slope form of the line: $y - y_0 = m(x - x_0)$

Plug in the slope: $y - y_0 = 6(x - x_0)$

Plug in the point (200, 2500): $y - 2500 = 6(x - 200)$

To find Anne's base salary, we plug in $x = 0$ and get $y - 2500 = -1200 \Rightarrow y = \1300 .

Anne's monthly base salary is \$1300.

Solving Real-World Problems Using Linear Models in Standard Form

Here are two examples of real-world problems where the standard form of an equation is useful.

Example 6

Nadia buys fruit at her local farmer's market. This Saturday, oranges cost \$2 per pound and cherries cost \$3 per pound. She has \$12 to spend on fruit. Write an equation in standard form that describes this situation. If she buys 4 pounds of oranges, how many pounds of cherries can she buy?

Solution

Let's define our variables:

x = pounds of oranges

y = pounds of cherries

The equation that describes this situation is $2x + 3y = 12$.

If she buys 4 pounds of oranges, we can plug $x = 4$ into the equation and solve for y :

$$2(4) + 3y = 12 \Rightarrow 3y = 12 - 8 \Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$$

Nadia can buy $1\frac{1}{3}$ pounds of cherries.

Example 7

Peter skateboards part of the way to school and walks the rest of the way. He can skateboard at 7 miles per hour and he can walk at 3 miles per hour. The distance to school is 6 miles. Write an equation in standard form that describes this situation. If he skateboards for $\frac{1}{2}$ an hour, how long does he need to walk to get to school?

Solution

Let's define our variables:

x = time Peter skateboards

y = time Peter walks

The equation that describes this situation is: $7x + 3y = 6$

If Peter skateboards $\frac{1}{2}$ an hour, we can plug $x = 0.5$ into the equation and solve for y :

$$7(0.5) + 3y = 6 \Rightarrow 3y = 6 - 3.5 \Rightarrow 3y = 2.5 \Rightarrow y = \frac{5}{6}$$

Peter must walk $\frac{5}{6}$ of an hour.

Further Practice

Now that you've worked with equations in all three basic forms, check out the Java applet at <http://www.ronblond.com/M10/lineAP/index.html>. You can use it to manipulate graphs of equations in all three forms, and see how the graphs change when you vary the terms of the equations.

Another applet at <http://www.cut-the-knot.org/Curriculum/Calculus/StraightLine.shtml> lets you create multiple lines and see how they intersect. Each line is defined by two points; you can change the slope of a line by moving either of the points, or just drag the whole line around without changing its slope. To create another line, just click Duplicate and then drag one of the lines that are already there.

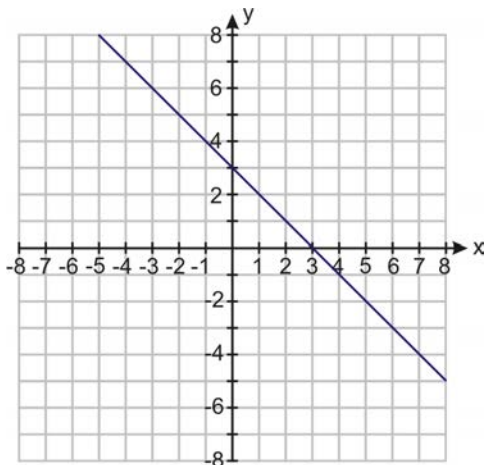
Review Questions

Find the equation of each line in slope-intercept form.

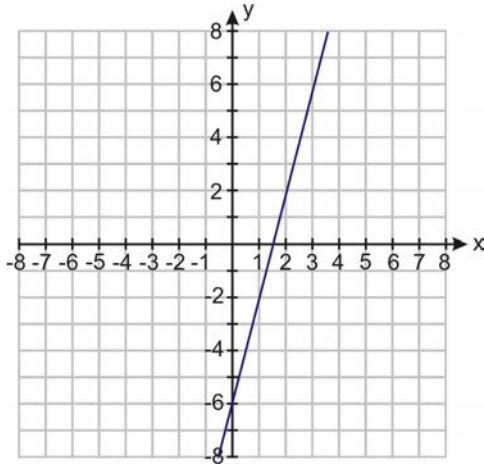
1. The line has a slope of 7 and a y -intercept of -2.
2. The line has a slope of -5 and a y -intercept of 6.
3. The line has a slope of $-\frac{1}{4}$ and contains the point $(4, -1)$.
4. The line has a slope of $\frac{2}{3}$ and contains the point $(\frac{1}{2}, 1)$.
5. The line has a slope of -1 and contains the point $(\frac{4}{5}, 0)$.
6. The line contains points $(2, 6)$ and $(5, 0)$.
7. The line contains points $(5, -2)$ and $(8, 4)$.
8. The line contains points $(3, 5)$ and $(-3, 0)$.
9. The line contains points $(10, 15)$ and $(12, 20)$.

Write the equation of each line in slope-intercept form.

10.



11.



Find the equation of each linear function in slope-intercept form.

12. $m = 5, f(0) = -3$
13. $m = -7, f(2) = -1$
14. $m = \frac{1}{3}, f(-1) = \frac{2}{3}$
15. $m = 4.2, f(-3) = 7.1$
16. $f\left(\frac{1}{4}\right) = \frac{3}{4}, f(0) = \frac{5}{4}$
17. $f(1.5) = -3, f(-1) = 2$

Write the equation of each line in point-slope form.

18. The line has slope $-\frac{1}{10}$ and goes through the point $(10, 2)$.
19. The line has slope -75 and goes through the point $(0, 125)$.
20. The line has slope 10 and goes through the point $(8, -2)$.
21. The line goes through the points $(-2, 3)$ and $(-1, -2)$.
22. The line contains the points $(10, 12)$ and $(5, 25)$.
23. The line goes through the points $(2, 3)$ and $(0, 3)$.
24. The line has a slope of $\frac{3}{5}$ and a y -intercept of -3 .
25. The line has a slope of -6 and a y -intercept of 0.5 .

Write the equation of each linear function in point-slope form.

26. $m = -\frac{1}{5}$ and $f(0) = 7$
27. $m = -12$ and $f(-2) = 5$
28. $f(-7) = 5$ and $f(3) = -4$
29. $f(6) = 0$ and $f(0) = 6$
30. $m = 3$ and $f(2) = -9$
31. $m = -\frac{9}{5}$ and $f(0) = 32$

Rewrite the following equations in standard form.

32. $y = 3x - 8$
33. $y - 7 = -5(x - 12)$
34. $2y = 6x + 9$

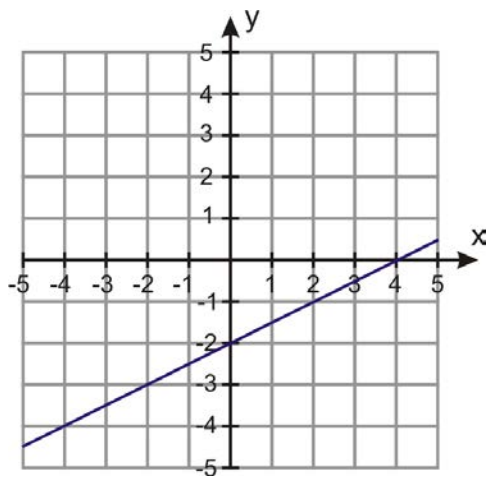
35. $y = \frac{9}{4}x + \frac{1}{4}$
 36. $y + \frac{2}{5} = \frac{2}{3}(x - 2)$
 37. $3y + 5 = 4(x - 9)$

Find the slope and y-intercept of the following lines.

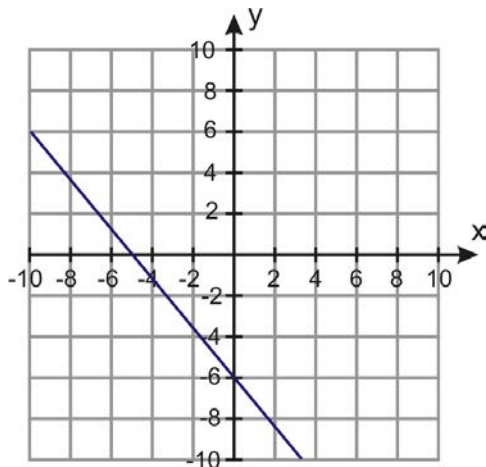
38. $5x - 2y = 15$
 39. $3x + 6y = 25$
 40. $x - 8y = 12$
 41. $3x - 7y = 20$
 42. $9x - 9y = 4$
 43. $6x + y = 3$

Find the equation of each line and write it in standard form.

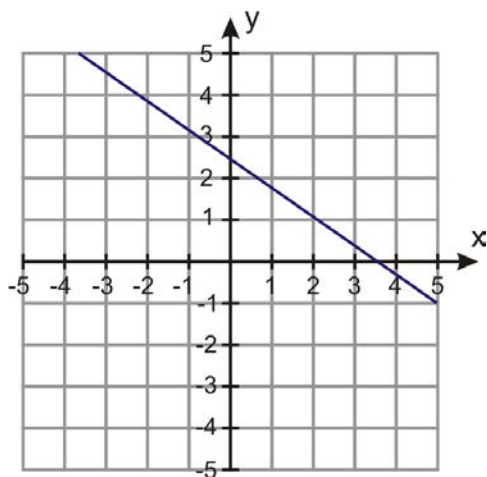
44.



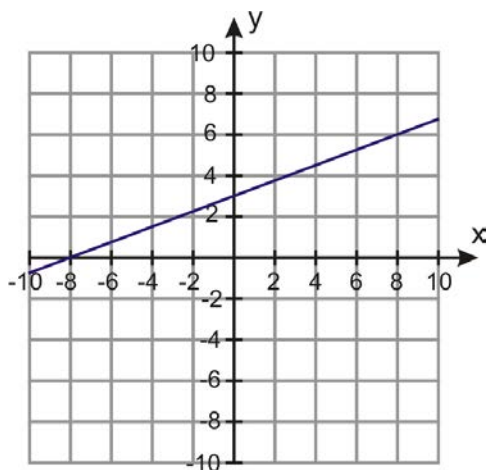
45.



46.



47.



48. Andrew has two part time jobs. One pays \$6 per hour and the other pays \$10 per hour. He wants to make \$366 per week. Write an equation in standard form that describes this situation. If he is only allowed to work 15 hours per week at the \$10 per hour job, how many hours does he need to work per week in his \$6 per hour job in order to achieve his goal?
49. Anne invests money in two accounts. One account returns 5% annual interest and the other returns 7% annual interest. In order not to incur a tax penalty, she can make no more than \$400 in interest per year. Write an equation in standard form that describes this problem. If she invests \$5000 in the 5% interest account, how much money does she need to invest in the other account?

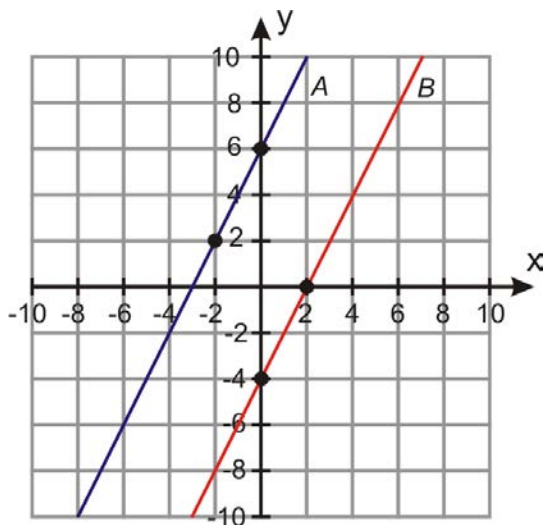
5.2 Equations of Parallel and Perpendicular Lines

Learning Objectives

- Determine whether lines are parallel or perpendicular
- Write equations of perpendicular lines
- Write equations of parallel lines
- Investigate families of lines

Introduction

In this section you will learn how **parallel lines** and **perpendicular lines** are related to each other on the coordinate plane. Let's start by looking at a graph of two parallel lines.



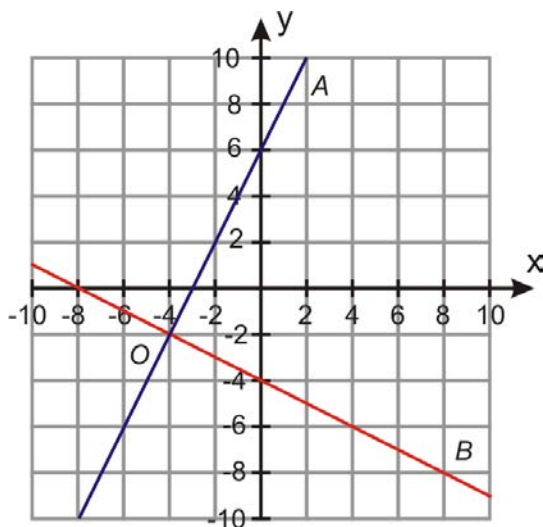
We can clearly see that the two lines have different y -intercepts: 6 and -4 .

How about the slopes of the lines? The slope of line A is $\frac{6-2}{0-(-2)} = \frac{4}{2} = 2$, and the slope of line B is $\frac{0-(-4)}{2-0} = \frac{4}{2} = 2$. The slopes are the same.

Is that significant? Yes. By definition, parallel lines never meet. That means that when one of them slopes up by a certain amount, the other one has to slope up by the same amount so the lines will stay the same distance apart. If you look at the graph above, you can see that for any x -value you pick, the y -values of lines A and B are the same vertical distance apart—which means that both lines go up by the same vertical distance every time they go across by the same horizontal distance. In order to stay parallel, their slopes must stay the same.

All parallel lines have the same slopes and different y -intercepts.

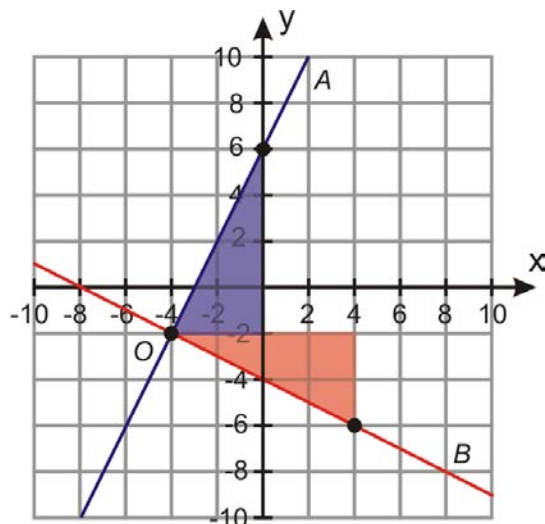
Now let's look at a graph of two perpendicular lines.



We can't really say anything about the y -intercepts. In this example, the y -intercepts are different, but if

we moved the lines four units to the right, they would both intercept the y -axis at $(0, -2)$. So perpendicular lines can have the same or different y -intercepts.

What about the relationship between the slopes of the two lines?



To find the slope of line A , we pick two points on the line and draw the blue (upper) right triangle. The legs of the triangle represent the rise and the run. We can see that the slope is $\frac{8}{4}$, or 2.

To find the slope of line B , we pick two points on the line and draw the red (lower) right triangle. Notice that the two triangles are identical, only rotated by 90° . Where line A goes 8 units up and 4 units right, line B goes 8 units right and 4 units down. Its slope is $-\frac{4}{8}$, or $-\frac{1}{2}$.

This is always true for perpendicular lines; where one line goes a units up and b units right, the other line will go a units right and b units down, so the slope of one line will be $\frac{a}{b}$ and the slope of the other line will be $-\frac{b}{a}$.

The slopes of **perpendicular lines** are always negative reciprocals of each other.

The Java applet at <http://members.shaw.ca/ron.blond/perp.APPLET/index.html> lets you drag around a pair of perpendicular lines to see how their slopes change. Click “Show Grid” to see the x - and y -axes, and click “Show Constructors” to see the triangles that are being used to calculate the slopes of the lines (you can then drag the circle to make it bigger or smaller, and click on a triangle to see the slope calculations in detail.)

Determine When Lines are Parallel or Perpendicular

You can find whether lines are parallel or perpendicular by comparing the slopes of the lines. If you are given points on the lines, you can find their slopes using the formula. If you are given the equations of the lines, re-write each equation in a form that makes it easy to read the slope, such as the slope-intercept form.

Example 1

Determine whether the lines are parallel or perpendicular or neither.

- One line passes through the points $(2, 11)$ and $(-1, 2)$; another line passes through the points $(0, -4)$ and $(-2, -10)$.
- One line passes through the points $(-2, -7)$ and $(1, 5)$; another line passes through the points $(4, 1)$ and $(-8, 4)$.

c) One line passes through the points (3, 1) and (-2, -2); another line passes through the points (5, 5) and (4, -6).

Solution

Find the slope of each line and compare them.

a) $m_1 = \frac{2-11}{-1-2} = \frac{-9}{-3} = 3$ and $m_2 = \frac{-10-(-4)}{-2-0} = \frac{-6}{-2} = 3$

The slopes are equal, so **the lines are parallel**.

b) $m_1 = \frac{5-(-7)}{1-(-2)} = \frac{12}{3} = 4$ and $m_2 = \frac{4-1}{-8-4} = \frac{3}{-12} = -\frac{1}{4}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular**.

c) $m_1 = \frac{-2-1}{-2-3} = \frac{-3}{-5} = \frac{3}{5}$ and $m_2 = \frac{-6-5}{4-5} = \frac{-13}{-1} = 13$

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular**.

Example 2

Determine whether the lines are parallel or perpendicular or neither:

a) $3x + 4y = 2$ and $8x - 6y = 5$

b) $2x = y - 10$ and $y = -2x + 5$

c) $7y + 1 = 7x$ and $x + 5 = y$

Solution

Write each equation in slope-intercept form:

a) line 1: $3x + 4y = 2 \Rightarrow 4y = -3x + 2 \Rightarrow y = -\frac{3}{4}x + \frac{1}{2} \Rightarrow$ slope = $-\frac{3}{4}$

line 2: $8x - 6y = 5 \Rightarrow 8x - 5 = 6y \Rightarrow y = \frac{8}{6}x - \frac{5}{6} \Rightarrow y = \frac{4}{3}x - \frac{5}{6} \Rightarrow$ slope = $\frac{4}{3}$

The slopes are negative reciprocals of each other, so **the lines are perpendicular**.

b) line 1: $2x = y - 10 \Rightarrow y = 2x + 10 \Rightarrow$ slope = 2

line 2: $y = -2x + 5 \Rightarrow$ slope = -2

The slopes are not the same or negative reciprocals of each other, so **the lines are neither parallel nor perpendicular**.

c) line 1: $7y + 1 = 7x \Rightarrow 7y = 7x - 1 \Rightarrow y = x - \frac{1}{7} \Rightarrow$ slope = 1

line 2: $x + 5 = y \Rightarrow y = x + 5 \Rightarrow$ slope = 1

The slopes are the same, so **the lines are parallel**.

Write Equations of Parallel and Perpendicular Lines

We can use the properties of parallel and perpendicular lines to write an equation of a line parallel or perpendicular to a given line. You might be given a line and a point, and asked to find the line that goes through the given point and is parallel or perpendicular to the given line. Here's how to do this:

1. Find the slope of the given line from its equation. (You might need to re-write the equation in a form such as the slope-intercept form.)
2. Find the slope of the parallel or perpendicular line—which is either the same as the slope you found in step 1 (if it's parallel), or the negative reciprocal of the slope you found in step 1 (if it's perpendicular).
3. Use the slope you found in step 2, along with the point you were given, to write an equation of the

new line in slope-intercept form or point-slope form.

Example 3

Find an equation of the line perpendicular to the line $y = -3x + 5$ that passes through the point $(2, 6)$.

Solution

The slope of the given line is -3 , so the perpendicular line will have a slope of $\frac{1}{3}$.

Now to find the equation of a line with slope $\frac{1}{3}$ that passes through $(2, 6)$:

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = \frac{1}{3}x + b$.

Plug in the point $(2, 6)$ to find b : $6 = \frac{1}{3}(2) + b \Rightarrow b = 6 - \frac{2}{3} \Rightarrow b = \frac{20}{3}$.

The equation of the line is $y = \frac{1}{3}x + \frac{20}{3}$.

Example 4

Find the equation of the line perpendicular to $x - 5y = 15$ that passes through the point $(-2, 5)$.

Solution

Re-write the equation in slope-intercept form: $x - 5y = 15 \Rightarrow -5y = -x + 15 \Rightarrow y = \frac{1}{5}x - 3$.

The slope of the given line is $\frac{1}{5}$, so we're looking for a line with slope -5 .

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = -5x + b$.

Plug in the point $(-2, 5)$: $5 = -5(-2) + b \Rightarrow b = 5 - 10 \Rightarrow b = -5$

The equation of the line is $y = -5x - 5$.

Example 5

Find the equation of the line parallel to $6x - 5y = 12$ that passes through the point $(-5, -3)$.

Solution

Rewrite the equation in slope-intercept form: $6x - 5y = 12 \Rightarrow 5y = 6x - 12 \Rightarrow y = \frac{6}{5}x - \frac{12}{5}$.

The slope of the given line is $\frac{6}{5}$, so we are looking for a line with slope $\frac{6}{5}$ that passes through the point $(-5, -3)$.

Start with the slope-intercept form: $y = mx + b$.

Plug in the slope: $y = \frac{6}{5}x + b$.

Plug in the point $(-5, -3)$: $-3 = \frac{6}{5}(-5) + b \Rightarrow -3 = -6 + b \Rightarrow b = 3$

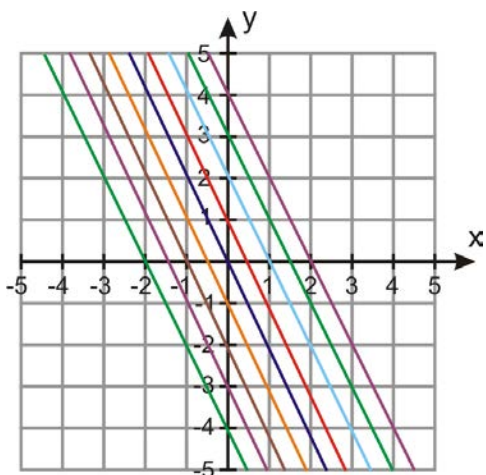
The equation of the line is $y = \frac{6}{5}x + 3$.

Investigate Families of Lines

A **family of lines** is a set of lines that have something in common with each other. Straight lines can belong to two types of families: one where the slope is the same and one where the y -intercept is the same.

Family 1: Keep the slope unchanged and vary the y -intercept.

The figure below shows the family of lines with equations of the form $y = -2x + b$:

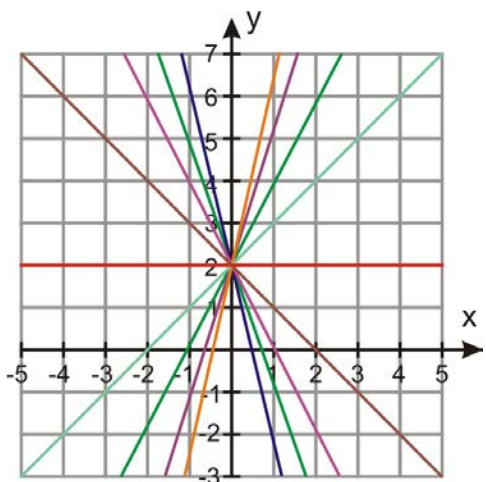


All the lines have a slope of -2 , but the value of b is different for each line.

Notice that in such a family all the lines are parallel. All the lines look the same, except that they are shifted up and down the y -axis. As b gets larger the line rises on the y -axis, and as b gets smaller the line goes lower on the y -axis. This behavior is often called a **vertical shift**.

Family 2: Keep the y -intercept unchanged and vary the slope.

The figure below shows the family of lines with equations of the form $y = mx + 2$:



All the lines have a y -intercept of two, but the slope is different for each line. The steeper lines have higher values of m .

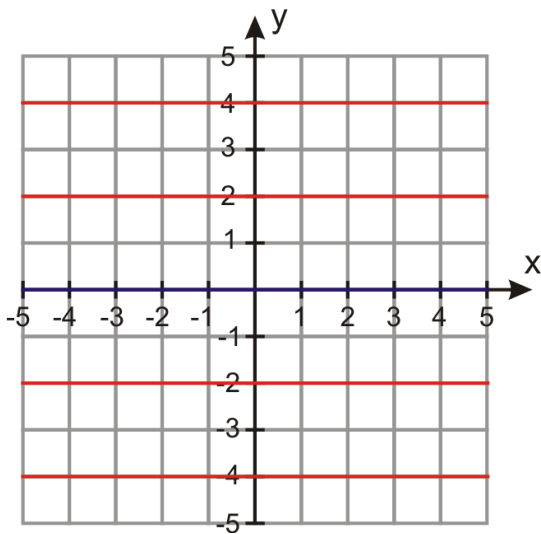
Example 6

Write the equation of the family of lines satisfying the given condition.

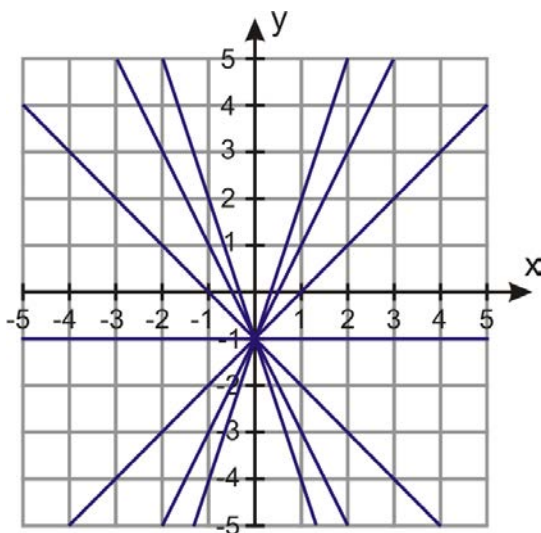
- parallel to the x -axis
- through the point $(0, -1)$
- perpendicular to $2x + 7y - 9 = 0$
- parallel to $x + 4y - 12 = 0$

Solution

a) All lines parallel to the x -axis have a slope of zero; the y -intercept can be anything. So the family of lines is $y = 0x + b$ or just $y = b$.

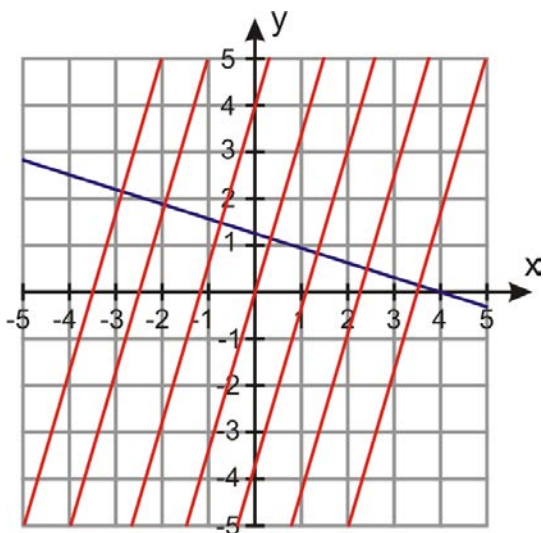


b) All lines passing through the point $(0, -1)$ have the same y -intercept, $b = -1$. The family of lines is: $y = mx - 1$.



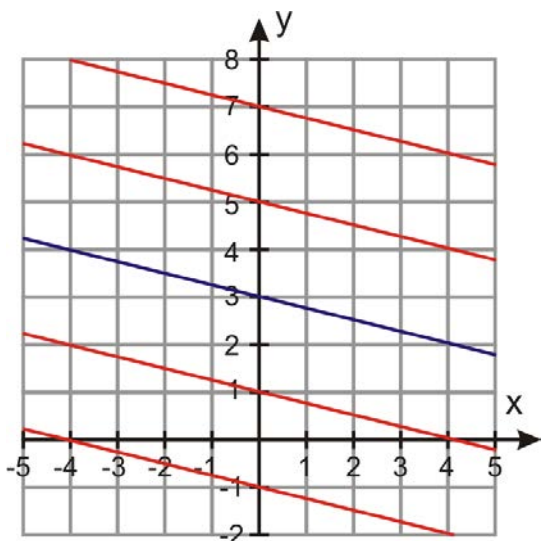
c) First we need to find the slope of the given line. Rewriting $2x + 7y - 9 = 0$ in slope-intercept form, we get $y = -\frac{2}{7}x + \frac{9}{7}$. The slope of the line is $-\frac{2}{7}$, so we're looking for the family of lines with slope $\frac{7}{2}$.

The family of lines is $y = \frac{7}{2}x + b$.



d) Rewrite $x + 4y - 12 = 0$ in slope-intercept form: $y = -\frac{1}{4}x + 3$. The slope is $-\frac{1}{4}$, so that's also the slope of the family of lines we are looking for.

The family of lines is $y = -\frac{1}{4}x + b$.



Review Questions

For questions 1-10, determine whether the lines are parallel, perpendicular or neither.

1. One line passes through the points $(-1, 4)$ and $(2, 6)$; another line passes through the points $(2, -3)$ and $(8, 1)$.
2. One line passes through the points $(4, -3)$ and $(-8, 0)$; another line passes through the points $(-1, -1)$ and $(-2, 6)$.
3. One line passes through the points $(-3, 14)$ and $(1, -2)$; another line passes through the points $(0, -3)$ and $(-2, 5)$.
4. One line passes through the points $(3, 3)$ and $(-6, -3)$; another line passes through the points $(2, -8)$ and $(-6, 4)$.
5. One line passes through the points $(2, 8)$ and $(6, 0)$; another line has the equation $x - 2y = 5$.

6. One line passes through the points $(-5, 3)$ and $(2, -1)$; another line has the equation $2x + 3y = 6$.
7. Both lines pass through the point $(2, 8)$; one line also passes through $(3, 5)$, and the other line has slope 3.
8. Line 1: $4y + x = 8$ Line 2: $12y + 3x = 1$
9. Line 1: $5y + 3x = 1$ Line 2: $6y + 10x = -3$
10. Line 1: $2y - 3x + 5 = 0$ Line 2: $y + 6x = -3$
11. Lines $A, B, C, D,$ and E all pass through the point $(3, 6)$. Line A also passes through $(7, 12)$; line B passes through $(8, 4)$; line C passes through $(-1, -3)$; line D passes through $(1, 1)$; and line E passes through $(6, 12)$.
 - (a) Are any of these lines perpendicular? If so, which ones? If not, why not?
 - (b) Are any of these lines parallel? If so, which ones? If not, why not?
12. Find the equation of the line parallel to $5x - 2y = 2$ that passes through point $(3, -2)$.
13. Find the equation of the line perpendicular to $y = -\frac{2}{5}x - 3$ that passes through point $(2, 8)$.
14. Find the equation of the line parallel to $7y + 2x - 10 = 0$ that passes through the point $(2, 2)$.
15. Find the equation of the line perpendicular to $y + 5 = 3(x - 2)$ that passes through the point $(6, 2)$.
16. Line S passes through the points $(2, 3)$ and $(4, 7)$. Line T passes through the point $(2, 5)$. If Lines S and T are parallel, name one more point on line T . (**Hint:** you don't need to find the slope of either line.)
17. Lines P and Q both pass through $(-1, 5)$. Line P also passes through $(-3, -1)$. If P and Q are perpendicular, name one more point on line Q . (This time you will have to find the slopes of both lines.)
18. Write the equation of the family of lines satisfying the given condition.
 - (a) All lines that pass through point $(0, 4)$.
 - (b) All lines that are perpendicular to $4x + 3y - 1 = 0$.
 - (c) All lines that are parallel to $y - 3 = 4x + 2$.
 - (d) All lines that pass through the point $(0, -1)$.
19. Name two lines that pass through the point $(3, -1)$ and are perpendicular to each other.
20. Name two lines that are each perpendicular to $y = -4x - 2$. What is the relationship of those two lines to each other?
21. Name two perpendicular lines that both pass through the point $(3, -2)$. Then name a line parallel to one of them that passes through the point $(-2, 5)$.

5.3 Fitting a Line to Data

Learning Objectives

- Make a scatter plot.
- Fit a line to data and write an equation for that line.
- Perform linear regression with a graphing calculator.
- Solve real-world problems using linear models of scattered data.

Introduction

Katja has noticed that sales are falling off at her store lately. She plots her sales figures for each week on a graph and sees that the points are trending downward, but they don't quite make a straight line. How can she predict what her sales figures will be over the next few weeks?

In real-world problems, the relationship between our dependent and independent variables is linear, but not perfectly so. We may have a number of data points that don't quite fit on a straight line, but we may still want to find an equation representing those points. In this lesson, we'll learn how to find linear equations to fit real-world data.

Make a Scatter Plot

A **scatter plot** is a plot of all the ordered pairs in a table. Even when we expect the relationship we're analyzing to be linear, we usually can't expect that all the points will fit perfectly on a straight line. Instead, the points will be "scattered" about a straight line.

There are many reasons why the data might not fall perfectly on a line. Small errors in measurement are one reason; another reason is that the real world isn't always as simple as a mathematical abstraction, and sometimes math can only describe it approximately.

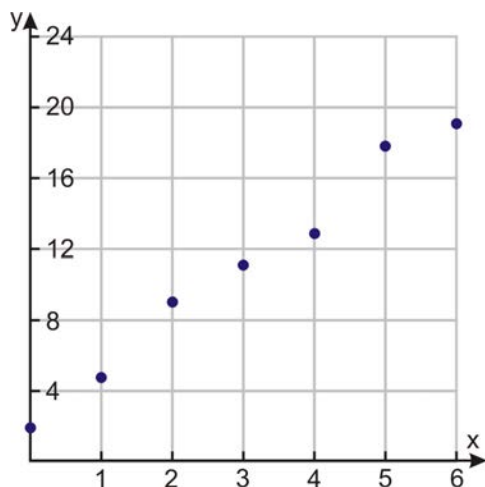
Example 1

Make a scatter plot of the following ordered pairs:

$(0, 2)$; $(1, 4.5)$; $(2, 9)$; $(3, 11)$; $(4, 13)$; $(5, 18)$; $(6, 19.5)$

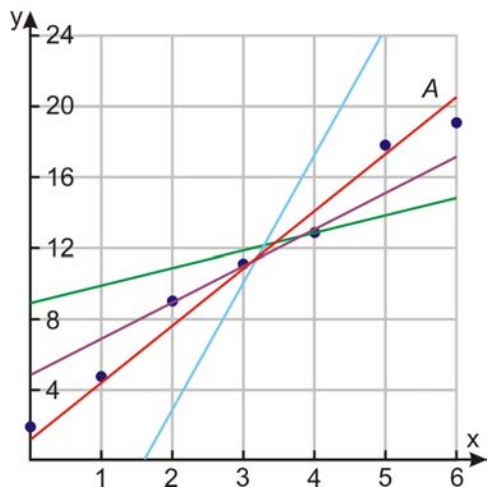
Solution

We make a scatter plot by graphing all the ordered pairs on the coordinate axis:

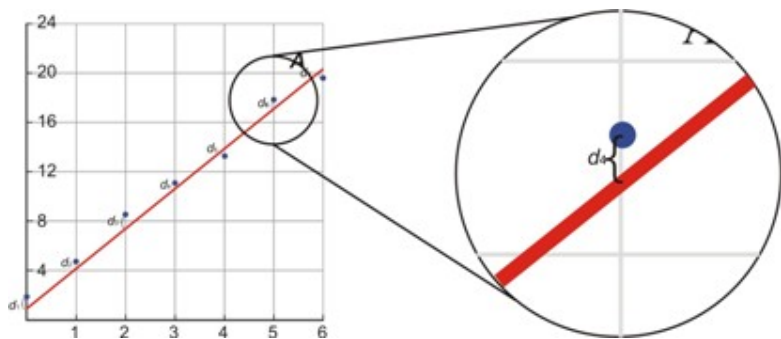


Fit a Line to Data

Notice that the points look like they might be part of a straight line, although they wouldn't fit perfectly on a straight line. If the points were perfectly lined up, we could just draw a line through any two of them, and that line would go right through all the other points as well. When the points aren't lined up perfectly, we just have to find a line that is as close to all the points as possible.



Here you can see that we could draw many lines through the points in our data set. However, the red line A is the line that best fits the points. To prove this mathematically, we would measure all the distances from each data point to line A: and then we would show that the sum of all those distances—or rather, the square root of the sum of the squares of the distances—is less than it would be for any other line.



Actually proving this is a lesson for a much more advanced course, so we won't do it here. And finding the best fit line in the first place is even more complex; instead of doing it by hand, we'll use a graphing calculator or just “eyeball” the line, as we did above—using our visual sense to guess what line fits best.

For more practice eyeballing lines of best fit, try the Java applet at <http://mste.illinois.edu/activity/regression/>. Click on the green field to place up to 50 points on it, then use the slider to adjust the slope of the red line to try and make it fit the points. (The thermometer shows how far away the line is from the points, so you want to try to make the thermometer reading as *low* as possible.) Then click “Show Best Fit” to show the actual best fit line in blue. Refresh the page or click “Reset” if you want to try again. For more of a challenge, try scattering the points in a less obvious pattern.

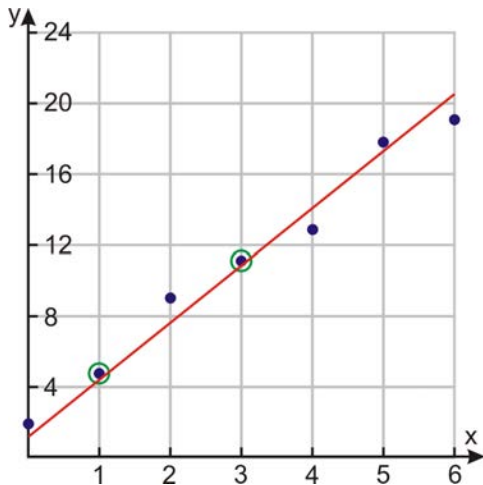
Write an Equation For a Line of Best Fit

Once you draw the line of best fit, you can find its equation by using two points on the line. Finding the equation of the line of best fit is also called **linear regression**.

Caution: Make sure you don't get caught making a common mistake. Sometimes the line of best fit won't pass straight through any of the points in the original data set. This means that you can't just use two points from the data set – **you need to use two points that are on the line**, which might not be in the data set at all.

In Example 1, it happens that two of the data points *are* very close to the line of best fit, so we can just

use these points to find the equation of the line: (1, 4.5) and (3, 11).



Start with the slope-intercept form of a line: $y = mx + b$

Find the slope: $m = \frac{11-4.5}{3-1} = \frac{6.5}{2} = 3.25$.

So $y = 3.25x + b$.

Plug (3, 11) into the equation: $11 = 3.25(3) + b \Rightarrow b = 1.25$

So the equation for the line that fits the data best is $y = 3.25x + 1.25$.

Perform Linear Regression With a Graphing Calculator

The problem with eyeballing a line of best fit, of course, is that you can't be sure how accurate your guess is. To get the most accurate equation for the line, we can use a graphing calculator instead. The calculator uses a mathematical algorithm to find the line that minimizes the sum of the squares.

Example 2

Use a graphing calculator to find the equation of the line of best fit for the following data:

(3, 12), (8, 20), (1, 7), (10, 23), (5, 18), (8, 24), (11, 30), (2, 10)

Solution

Step 1: Input the data in your calculator.

Press [STAT] and choose the [EDIT] option. Input the data into the table by entering the x -values in the first column and the y -values in the second column.

L1	L2	L3	2
1	7		
10	23		
5	18		
8	24		
11	30		
2	10		

L2(8) = 10			

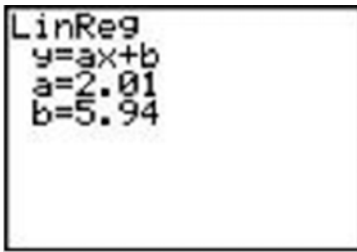
Step 2: Find the equation of the line of best fit.

Press [STAT] again use right arrow to select [CALC] at the top of the screen.

Chose option number 4, $LinReg(ax + b)$, and press [ENTER]

The calculator will display $LinReg(ax + b)$.

Press [ENTER] and you will be given the a - and b -values.



```
LinReg
y=ax+b
a=2.01
b=5.94
```

Here a represents the slope and b represents the y -intercept of the equation. The linear regression line is $y = 2.01x + 5.94$.

Step 3. Draw the scatter plot.

To draw the scatter plot press [STATPLOT] [2nd] [Y=].



```
Plot1 Plot2 Plot3
Off Off Off
Type: [Scatter] [Line] [Bar]
Xlist: L1
Ylist: L2
Mark: [Square] [Dot] [Cross]
```

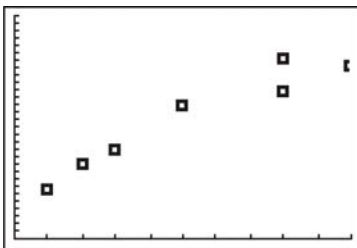
Choose Plot 1 and press [ENTER].

Press the On option and set the Type as scatter plot (the one highlighted in black).

Make sure that the X list and Y list names match the names of the columns of the table in Step 1.

Choose the box or plus as the mark, since the simple dot may make it difficult to see the points.

Press [GRAPH] and adjust the window size so you can see all the points in the scatter plot.

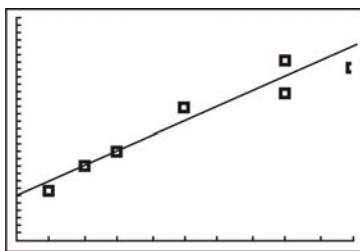


Step 4. Draw the line of best fit through the scatter plot.

Press [Y=]

Enter the equation of the line of best fit that you just found: $y = 2.01x + 5.94$.

Press [GRAPH].



Solve Real-World Problems Using Linear Models of Scattered Data

Once we've found the line of best fit for a data set, we can use the equation of that line to predict other data points.

Example 3

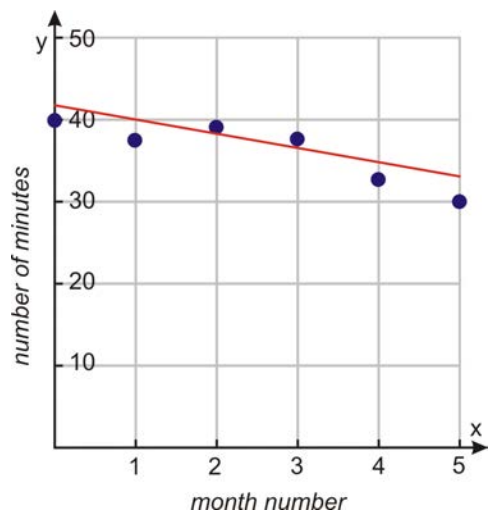
Nadia is training for a 5K race. The following table shows her times for each month of her training program. Find an equation of a line of fit. Predict her running time if her race is in August.

Table 5.1:

Month	Month number	Average time (minutes)
January	0	40
February	1	38
March	2	39
April	3	38
May	4	33
June	5	30

Solution

Let's make a scatter plot of Nadia's running times. The independent variable, x , is the month number and the dependent variable, y , is the running time. We plot all the points in the table on the coordinate plane, and then sketch a line of fit.



Two points on the line are (0, 42) and (4, 34). We'll use them to find the equation of the line:

$$\begin{aligned}m &= \frac{34 - 42}{4 - 0} = -\frac{8}{4} = -2 \\y &= -2x + b \\42 &= -2(0) + b \Rightarrow b = 42 \\y &= -2x + 42\end{aligned}$$

In a real-world problem, the slope and y -intercept have a physical significance. In this case, the slope tells us how Nadia's running time changes each month she trains. Specifically, it decreases by 2 minutes per month. Meanwhile, the y -intercept tells us that when Nadia started training, she ran a distance of 5K in 42 minutes.

The problem asks us to predict Nadia's running time in August. Since June is defined as month number 5, August will be month number 7. We plug $x = 7$ into the equation of the line of best fit:

$$y = -2(7) + 42 = -14 + 42 = 28$$

The equation predicts that **Nadia will run the 5K race in 28 minutes.**

In this solution, we eyeballed a line of fit. Using a graphing calculator, we can find this equation for a line of fit instead: $y = -2.2x + 43.7$

If we plug $x = 7$ into this equation, we get $y = -2.2(7) + 43.7 = 28.3$. This means that **Nadia will run her race in 28.3 minutes.** You see that the graphing calculator gives a different equation and a different answer to the question. The graphing calculator result is more accurate, but the line we drew by hand still gives a good approximation to the result. And of course, there's no guarantee that Nadia will actually finish the race in that exact time; both answers are estimates, it's just that the calculator's estimate is slightly more likely to be right.

Example 4

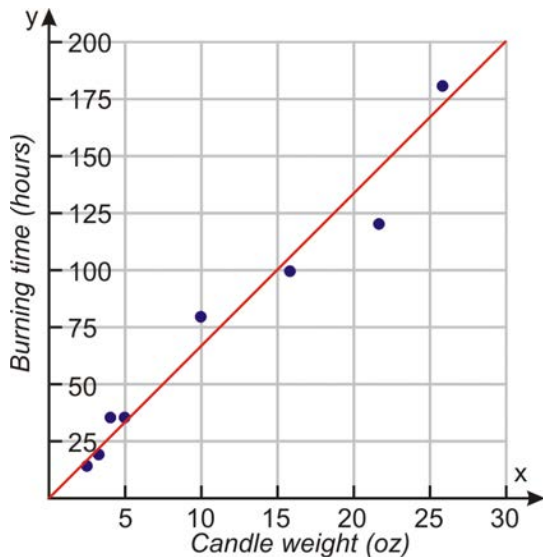
Peter is testing the burning time of "BriteGlo" candles. The following table shows how long it takes to burn candles of different weights. Assume it's a linear relation, so we can use a line to fit the data. If a candle burns for 95 hours, what must be its weight in ounces?

Table 5.2:

Candle weight (oz)	Time (hours)
2	15
3	20
4	35
5	36
10	80
16	100
22	120
26	180

Solution

Let's make a scatter plot of the data. The independent variable, x , is the candle weight and the dependent variable, y , is the time it takes the candle to burn. We plot all the points in the table on the coordinate plane, and draw a line of fit.



Two convenient points on the line are $(0,0)$ and $(30, 200)$. Find the equation of the line:

$$m = \frac{200}{30} = \frac{20}{3}$$

$$y = \frac{20}{3}x + b$$

$$0 = \frac{20}{3}(0) + b \Rightarrow b = 0$$

$$y = \frac{20}{3}x$$

A slope of $\frac{20}{3} = 6\frac{2}{3}$ tells us that for each extra ounce of candle weight, the burning time increases by $6\frac{2}{3}$ hours. A y -intercept of zero tells us that a candle of weight 0 oz will burn for 0 hours.

The problem asks for the weight of a candle that burns 95 hours; in other words, what's the x -value that gives a y -value of 95? Plugging in $y = 95$:

$$y = \frac{20}{3}x \Rightarrow 95 = \frac{20}{3}x \Rightarrow x = \frac{285}{20} = \frac{57}{4} = 14\frac{1}{4}$$

A candle that burns 95 hours weighs 14.25 oz.

A graphing calculator gives the linear regression equation as $y = 6.1x + 5.9$ and a result of **14.6 oz**.

Review Questions

For problems 1-4, draw the scatter plot and find an equation that fits the data set by hand.

- $(57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)$
- $(32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (39, 48)$
- $(12, 18); (5, 24); (15, 16); (11, 19); (9, 12); (7, 13); (6, 17); (12, 14)$
- $(3, 12); (8, 20); (1, 7); (10, 23); (5, 18); (8, 24); (2, 10)$
- Use the graph from problem 1 to predict the y -values for two x -values of your choice that are not in the data set.
- Use the graph from problem 2 to predict the x -values for two y -values of your choice that are not in the data set.

7. Use the equation from problem 3 to predict the y -values for two x -values of your choice that are not in the data set.
8. Use the equation from problem 4 to predict the x -values for two y -values of your choice that are not in the data set.

For problems 9-11, use a graphing calculator to find the equation of the line of best fit for the data set.

9. (57, 45); (65, 61); (34, 30); (87, 78); (42, 41); (35, 36); (59, 35); (61, 57); (25, 23); (35, 34)
10. (32, 43); (54, 61); (89, 94); (25, 34); (43, 56); (58, 67); (38, 46); (47, 56); (95, 105); (39, 48)
11. (12, 18); (3, 26); (5, 24); (15, 16); (11, 19); (0, 27); (9, 12); (7, 13); (6, 17); (12, 14)
12. Graph the best fit line on top of the scatter plot for problem 10. Then pick a data point that's close to the line, and change its y -value to move it much farther from the line.
 - (a) Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
 - (b) How much did the slope of the best fit line change when you changed that point?
13. Graph the scatter plot from problem 11 and change one point as you did in the previous problem.
 - (a) Calculate the new best fit line with that one point changed; write the equation of that line along with the coordinates of the new point.
 - (b) Did changing that one point seem to affect the slope of the best fit line more or less than it did in the previous problem? What might account for this difference?
14. Shiva is trying to beat the samosa-eating record. The current record is 53.5 samosas in 12 minutes. Each day he practices and the following table shows how many samosas he eats each day for the first week of his training.

Table 5.3:

Day	No. of samosas
1	30
2	34
3	36
4	36
5	40
6	43
7	45

- (a) Draw a scatter plot and find an equation to fit the data.
 - (b) Will he be ready for the contest if it occurs two weeks from the day he started training?
 - (c) What are the meanings of the slope and the y -intercept in this problem?
15. Anne is trying to find the elasticity coefficient of a Superball. She drops the ball from different heights and measures the maximum height of the ball after the bounce. The table below shows the data she collected.

Table 5.4:

Initial height (cm)	Bounce height (cm)
30	22
35	26
40	29
45	34
50	38
55	40
60	45
65	50
70	52

- (a) Draw a scatter plot and find the equation.
- (b) What height would she have to drop the ball from for it to bounce 65 cm?
- (c) What are the meanings of the slope and the y -intercept in this problem?
- (d) Does the y -intercept make sense? Why isn't it $(0, 0)$?

16. The following table shows the median California family income from 1995 to 2002 as reported by the US Census Bureau.

Table 5.5:

Year	Income
1995	53,807
1996	55,217
1997	55,209
1998	55,415
1999	63,100
2000	63,206
2001	63,761
2002	65,766

- (a) Draw a scatter plot and find the equation.
- (b) What would you expect the median annual income of a Californian family to be in year 2010?
- (c) What are the meanings of the slope and the y -intercept in this problem?
- (d) Inflation in the U.S. is measured by the Consumer Price Index, which increased by 20% between 1995 and 2002. Did the median income of California families keep up with inflation over that time period? (In other words, did it increase by at least 20%?)

5.4 Predicting with Linear Models

Learning Objectives

- Interpolate using an equation.
- Extrapolate using an equation.
- Predict using an equation.

Introduction

Katja’s sales figures were trending downward quickly at first, and she used a line of best fit to describe the numbers. But now they seem to be decreasing more slowly, and fitting the line less and less accurately. How can she make a more accurate prediction of what next week’s sales will be?

In the last lesson we saw how to find the equation of a line of best fit and how to use this equation to make predictions. The line of “best fit” is a good method if the relationship between the dependent and the independent variables is linear. In this section you will learn other methods that are useful even when the relationship isn’t linear.

Linear Interpolation

We use linear interpolation to fill in gaps in our data—that is, to estimate values that fall in between the values we already know. To do this, we use a straight line to connect the known data points on either side of the unknown point, and use the equation of that line to estimate the value we are looking for.

Example 1

The following table shows the median ages of first marriage for men and women, as gathered by the U.S. Census Bureau.

Table 5.6:

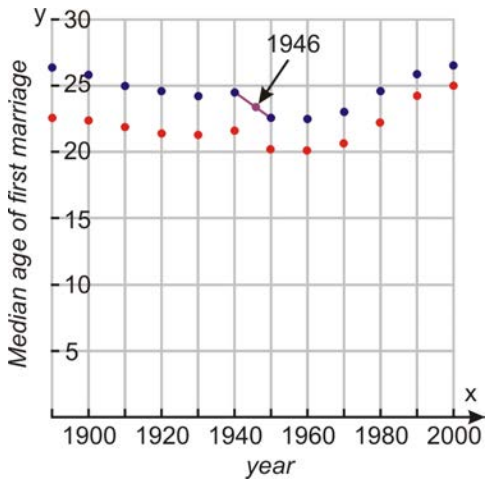
Year	Median age of males	Median age of females
1890	26.1	22.0
1900	25.9	21.9
1910	25.1	21.6
1920	24.6	21.2
1930	24.3	21.3
1940	24.3	21.5
1950	22.8	20.3
1960	22.8	20.3
1970	23.2	20.8
1980	24.7	22.0
1990	26.1	23.9
2000	26.8	25.1

Estimate the median age for the first marriage of a male in the year 1946.

Solution

We connect the two points on either side of 1946 with a straight line and find its equation. Here’s how

that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$m = \frac{22.8 - 24.3}{1950 - 1940} = \frac{-1.5}{10} = -0.15$$

$$y = -0.15x + b$$

$$24.3 = -0.15(1940) + b$$

$$b = 315.3$$

Our equation is $y = -0.15x + 315.3$.

To estimate the median age of marriage of males in the year 1946, we plug $x = 1946$ into the equation we just found:

$$y = -0.15(1946) + 315.3 = 23.4 \text{ years old}$$

Example 2

The Center for Disease Control collects information about the health of the American people and behaviors that might lead to bad health. The following table shows the percent of women who smoke during pregnancy.

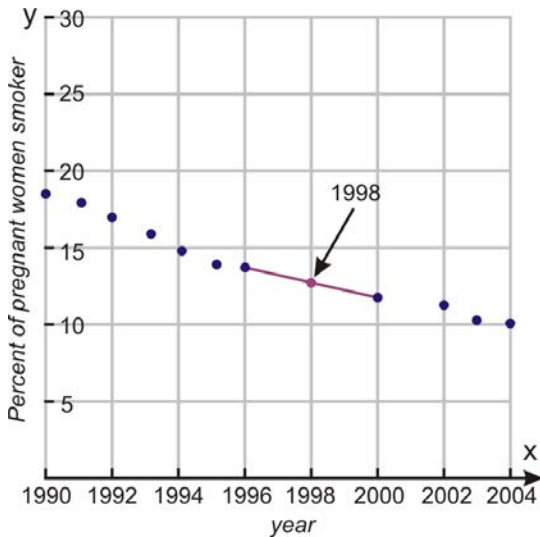
Table 5.7:

Year	Percent of pregnant women smokers
1990	18.4
1991	17.7
1992	16.9
1993	15.8
1994	14.6
1995	13.9
1996	13.6
2000	12.2
2002	11.4
2003	10.4
2004	10.2

Estimate the percentage of pregnant women that were smoking in the year 1998.

Solution

We connect the two points on either side of 1998 with a straight line and find its equation. Here's how that looks on a scatter plot:



We find the equation by plugging in the two data points:

$$m = \frac{12.2 - 13.6}{2000 - 1996} = \frac{-1.4}{4} = -0.35$$
$$y = -0.35x + b$$
$$12.2 = -0.35(2000) + b$$
$$b = 712.2$$

Our equation is $y = -0.35x + 712.2$.

To estimate the percentage of pregnant women who smoked in the year 1998, we plug $x = 1998$ into the equation we just found:

$$y = -0.35(1998) + 712.2 = 12.9\%$$

For non-linear data, linear interpolation is often not accurate enough for our purposes. If the points in the data set change by a large amount in the interval you're interested in, then linear interpolation may not give a good estimate. In that case, it can be replaced by **polynomial interpolation**, which uses a curve instead of a straight line to estimate values between points. But that's beyond the scope of this lesson.

Linear Extrapolation

Linear extrapolation can help us estimate values that are outside the range of our data set. The strategy is similar to linear interpolation: we pick the two data points that are closest to the one we're looking for, find the equation of the line between them, and use that equation to estimate the coordinates of the missing point.

Example 3

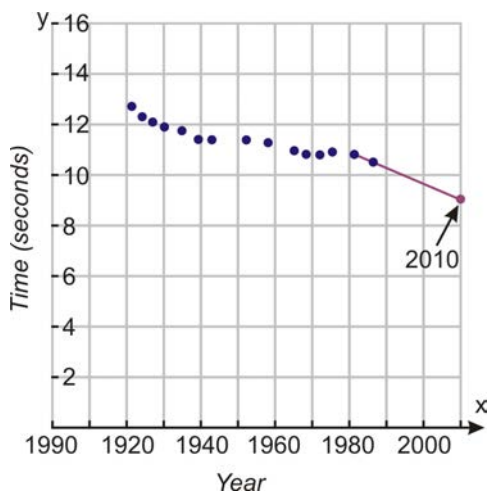
The winning times for the women's 100 meter race are given in the following table. Estimate the winning time in the year 2010. Is this a good estimate?

Table 5.8:

Winner	Country	Year	Time (seconds)
Mary Lines	UK	1922	12.8
Leni Schmidt	Germany	1925	12.4
Gerturd Glasitsch	Germany	1927	12.1
Tollien Schuurman	Netherlands	1930	12.0
Helen Stephens	USA	1935	11.8
Lulu Mae Hymes	USA	1939	11.5
Fanny Blankers-Koen	Netherlands	1943	11.5
Marjorie Jackson	Australia	1952	11.4
Vera Krepkina	Soviet Union	1958	11.3
Wyomia Tyus	USA	1964	11.2
Barbara Ferrell	USA	1968	11.1
Ellen Strophal	East Germany	1972	11.0
Inge Helten	West Germany	1976	11.0
Marlies Gohr	East Germany	1982	10.9
Florence Griffith Joyner	USA	1988	10.5

Solution

We start by making a scatter plot of the data; then we connect the last two points on the graph and find the equation of the line.



$$m = \frac{10.5 - 10.9}{1988 - 1982} = \frac{-0.4}{6} = -0.067$$

$$y = -0.067x + b$$

$$10.5 = -0.067(1988) + b$$

$$b = 143.7$$

Our equation is $y = -0.067x + 143.7$.

The winning time in year 2010 is estimated to be:

$$y = -0.067(2010) + 143.7 = 9.03 \text{ seconds.}$$

Unfortunately, this estimate actually isn't very accurate. This example demonstrates the weakness of linear extrapolation; it uses only a couple of points, instead of using *all* the points like the best fit line method, so

it doesn't give as accurate results when the data points follow a linear pattern. In this particular example, the last data point clearly doesn't fit in with the general trend of the data, so the slope of the extrapolation line is much steeper than it would be if we'd used a line of best fit. (As a historical note, the last data point corresponds to the winning time for Florence Griffith Joyner in 1988. After her race she was accused of using performance-enhancing drugs, but this fact was never proven. In addition, there was a question about the accuracy of the timing: some officials said that tail-wind was not accounted for in this race, even though all the other races of the day were affected by a strong wind.)

Here's an example of a problem where linear extrapolation does work better than the line of best fit method.

Example 4

A cylinder is filled with water to a height of 73 centimeters. The water is drained through a hole in the bottom of the cylinder and measurements are taken at 2 second intervals. The following table shows the height of the water level in the cylinder at different times.

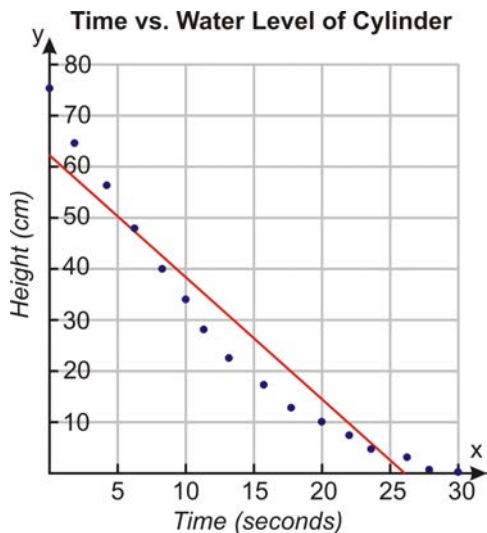
Table 5.9:

Time (seconds)	Water level (cm)
0.0	73
2.0	63.9
4.0	55.5
6.0	47.2
8.0	40.0
10.0	33.4
12.0	27.4
14.0	21.9
16.0	17.1
18.0	12.9
20.0	9.4
22.0	6.3
24.0	3.9
26.0	2.0
28.0	0.7
30.0	0.1

- Find the water level at time 15 seconds.*
- Find the water level at time 27 seconds*
- What would be the original height of the water in the cylinder if the water takes 5 extra seconds to drain? (Find the height at time of -5 seconds.)*

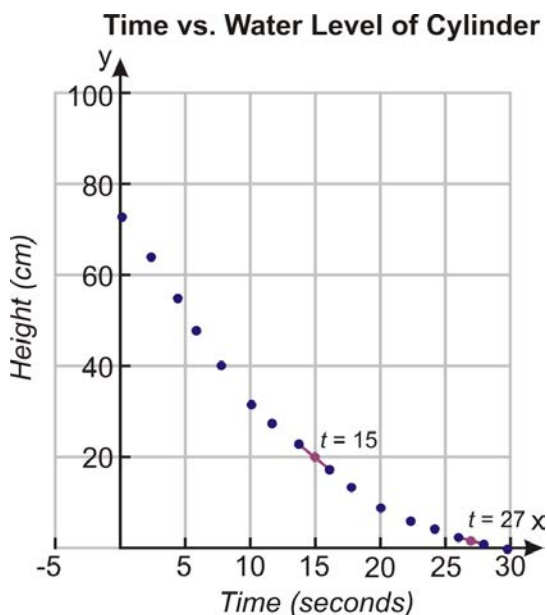
Solution

Here's what the line of best fit would look like for this data set:



Notice that the data points don't really make a line, and so the line of best fit still isn't a terribly good fit. Just a glance tells us that we'd estimate the water level at 15 seconds to be about 27 cm, which is *more* than the water level at 14 seconds. That's clearly not possible! Similarly, at 27 seconds we'd estimate the water to have all drained out, which it clearly hasn't yet.

So let's see what happens if we use linear extrapolation and interpolation instead. First, here are the lines we'd use to interpolate between 14 and 16 seconds, and between 26 and 28 seconds.



a) The slope of the line between points (14, 21.9) and (16, 17.1) is $m = \frac{17.1-21.9}{16-14} = \frac{-4.8}{2} = -2.4$. So $y = -2.4x + b \Rightarrow 21.9 = -2.4(14) + b \Rightarrow b = 55.5$, and the equation is $y = -2.4x + 55.5$.

Plugging in $x = 15$ gives us $y = -2.4(15) + 55.5 = 19.5$ cm.

b) The slope of the line between points (26, 2) and (28, 0.7) is $m = \frac{0.7-2}{28-26} = \frac{-1.3}{2} = -0.65$, so $y = -0.65x + b \Rightarrow 2 = -0.65(26) + b \Rightarrow b = 18.9$, and the equation is $y = -0.65x + 18.9$.

Plugging in $x = 27$, we get $y = -0.65(27) + 18.9 = 1.35$ cm.

c) Finally, we can use extrapolation to estimate the height of the water at -5 seconds. The slope of the line between points (0, 73) and (2, 63.9) is $m = \frac{63.9-73}{2-0} = \frac{-9.1}{2} = -4.55$, so the equation of the line is

$$y = -4.55x + 73.$$

Plugging in $x = -5$ gives us $y = -4.55(-5) + 73 = 95.75$ cm.

To make linear interpolation easier in the future, you might want to use the calculator at http://www.ajdesigner.com/phpinterpolation/linear_interpolation_equation.php. Plug in the coordinates of the first known data point in the blanks labeled x_1 and y_1 , and the coordinates of the second point in the blanks labeled x_3 and y_3 ; then enter the x -coordinate of the point in between in the blank labeled x_2 , and the y -coordinate will be displayed below when you click “Calculate.”

Review Questions

1. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1946. Fit a line, by hand, to the data before 1970.
2. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for females in 1984. Fit a line, by hand, to the data from 1970 on in order to estimate this accurately.
3. Use the data from Example 1 (*Median age at first marriage*) to estimate the age at marriage for males in 1995. Use linear interpolation between the 1990 and 2000 data points.
4. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 1997. Use linear interpolation between the 1996 and 2000 data points.
5. Use the data from Example 2 (*Pregnant women and smoking*) to estimate the percentage of pregnant smokers in 2006. Use linear extrapolation with the final two data points.
6. Use the data from Example 3 (*Winning times*) to estimate the winning time for the female 100-meter race in 1920. Use linear extrapolation because the first two or three data points have a different slope than the rest of the data.
7. The table below shows the highest temperature vs. the hours of daylight for the 15th day of each month in the year 2006 in San Diego, California.

Table 5.10:

Hours of daylight	High temperature (F)
10.25	60
11.0	62
12	62
13	66
13.8	68
14.3	73
14	86
13.4	75
12.4	71
11.4	66
10.5	73
10	61

- (a) What would be a better way to organize this table if you want to make the relationship between daylight hours and temperature easier to see?
- (b) Estimate the high temperature for a day with 13.2 hours of daylight using linear interpolation.
- (c) Estimate the high temperature for a day with 9 hours of daylight using linear extrapolation. Is the

prediction accurate?

(d) Estimate the high temperature for a day with 9 hours of daylight using a line of best fit.

The table below lists expected life expectancies based on year of birth (US Census Bureau). Use it to answer questions 8-15.

Table 5.11:

Birth year	Life expectancy in years
1930	59.7
1940	62.9
1950	68.2
1960	69.7
1970	70.8
1980	73.7
1990	75.4
2000	77

8. Make a scatter plot of the data.
9. Use a line of best fit to estimate the life expectancy of a person born in 1955.
10. Use linear interpolation to estimate the life expectancy of a person born in 1955.
11. Use a line of best fit to estimate the life expectancy of a person born in 1976.
12. Use linear interpolation to estimate the life expectancy of a person born in 1976.
13. Use a line of best fit to estimate the life expectancy of a person born in 2012.
14. Use linear extrapolation to estimate the life expectancy of a person born in 2012.
15. Which method gives better estimates for this data set? Why?

The table below lists the high temperature for the first day of the month for the year 2006 in San Diego, California (Weather Underground). Use it to answer questions 16-21.

Table 5.12:

Month number	Temperature (F)
1	63
2	66
3	61
4	64
5	71
6	78
7	88
8	78
9	81
10	75
11	68
12	69

16. Draw a scatter plot of the data.
17. Use a line of best fit to estimate the temperature in the middle of the 4th month (month 4.5).

18. Use linear interpolation to estimate the temperature in the middle of the 4th month (month 4.5).
19. Use a line of best fit to estimate the temperature for month 13 (January 2007).
20. Use linear extrapolation to estimate the temperature for month 13 (January 2007).
21. Which method gives better estimates for this data set? Why?
22. Name a real-world situation where you might want to make predictions based on available data. Would linear extrapolation/interpolation or the best fit method be better to use in that situation? Why?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9615>.

Chapter 6

Linear Inequalities

6.1 Solving Inequalities

Learning Objectives

- Write and graph inequalities in one variable on a number line.
- Solve inequalities using addition and subtraction.
- Solve inequalities using multiplication and division.
- Solve multi-step inequalities.

Introduction

Dita has a budget of \$350 to spend on a rental car for an upcoming trip, but she wants to spend as little of that money as possible. If the trip will last five days, what range of daily rental rates should she be willing to consider?

Like equations, inequalities show a relationship between two expressions. We solve and graph inequalities in a similar way to equations—but when we solve an inequality, the answer is usually a set of values instead of just one value.

When writing inequalities we use the following symbols:

$>$ is greater than

\geq is greater than or equal to

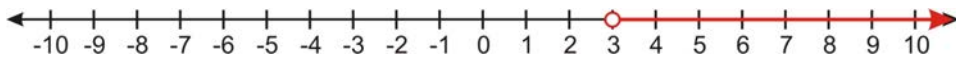
$<$ is less than

\leq is less than or equal to

Write and Graph Inequalities in One Variable on a Number Line

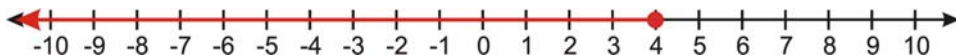
Let's start with the simple inequality $x > 3$.

We read this inequality as “ x is greater than 3.” The solution is the set of all real numbers that are greater than three. We often represent the solution set of an inequality with a number line graph.



Consider another simple inequality: $x \leq 4$.

We read this inequality as “ x is less than or equal to 4.” The solution is the set of all real numbers that are equal to four or less than four. We can graph this solution set on the number line.



Notice that we use an empty circle for the endpoint of a strict inequality (like $x > 3$), and a filled circle for one where the equals sign is included (like $x \leq 4$).

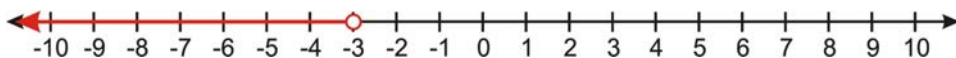
Example 1

Graph the following inequalities on the number line.

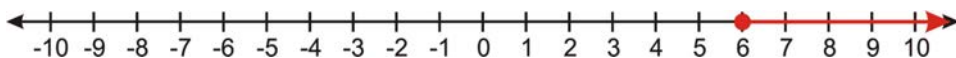
- $x < -3$
- $x \geq 6$
- $x > 0$
- $x \leq 8$

Solution

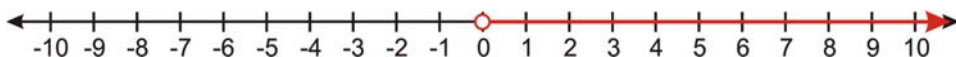
a) The inequality $x < -3$ represents all numbers that are less than -3 . The number -3 is not included in the solution, so it is represented by an open circle on the graph.



b) The inequality $x \geq 6$ represents all numbers that are greater than or equal to 6. The number 6 is included in the solution, so it is represented by a closed circle on the graph.



c) The inequality $x > 0$ represents all numbers that are greater than 0. The number 0 is not included in the solution, so it is represented by an open circle on the graph.

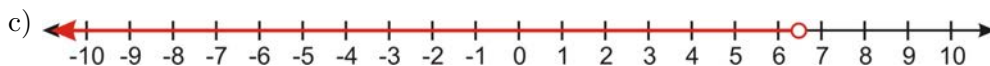
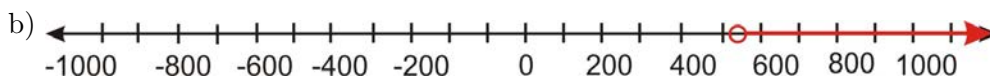
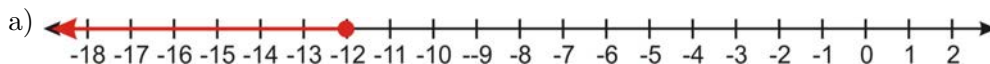


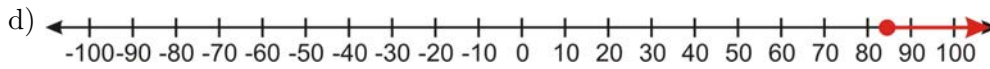
d) The inequality $x \leq 8$ represents all numbers that are less than or equal to 8. The number 8 is included in the solution, so it is represented by a closed circle on the graph.



Example 2

Write the inequality that is represented by each graph.





Solution

- a) $x \leq -12$
- b) $x > 540$
- c) $x < 6.5$
- d) $x \geq 85$

Inequalities appear everywhere in real life. Here are some simple examples of real-world applications.

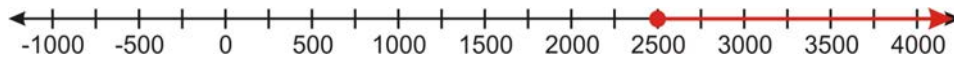
Example 3

Write each statement as an inequality and graph it on the number line.

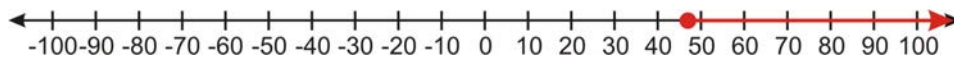
- a) You must maintain a balance of at least \$2500 in your checking account to get free checking.
- b) You must be at least 48 inches tall to ride the “Thunderbolt” Rollercoaster.
- c) You must be younger than 3 years old to get free admission at the San Diego Zoo.
- d) The speed limit on the interstate is 65 miles per hour or less.

Solution

a) The words “at least” imply that the value of \$2500 is included in the solution set, so the inequality is written as $x \geq 2500$.



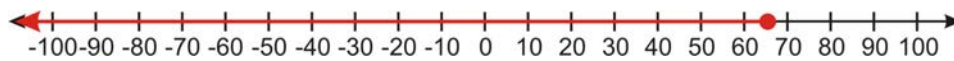
b) The words “at least” imply that the value of 48 inches is included in the solution set, so the inequality is written as $x \geq 48$.



c) The inequality is written as $x < 3$.



d) Speed limit means the highest allowable speed, so the inequality is written as $x \leq 65$.



Solving Inequalities Using Addition and Subtraction

To solve an inequality we must isolate the variable on one side of the inequality sign. To isolate the variable, we use the same basic techniques used in solving equations.

We can solve some inequalities by adding or subtracting a constant from one side of the inequality.

Example 4

Solve each inequality and graph the solution set.

- a) $x - 3 < 10$
- b) $x - 20 \geq 14$

c) $x + 8 \leq -7$

d) $x + 4 > 13$

Solution

a) Starting inequality: $x - 3 < 10$

Add **3** to both sides of the inequality: $x - 3 + 3 < 10 + 3$

Simplify: $x < 13$



b) Starting inequality: $x - 20 \leq 14$

Add **20** to both sides of the inequality: $x - 20 + 20 \leq 14 + 20$

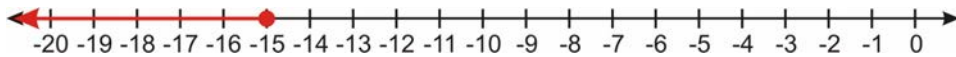
Simplify: $x \leq 34$



c) Starting inequality: $x + 8 \leq -7$

Subtract **8** from both sides of the inequality: $x + 8 - 8 \leq -7 - 8$

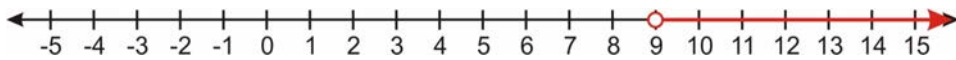
Simplify: $x \leq -15$



d) Starting inequality: $x + 4 > 13$

Subtract **4** from both sides of the inequality: $x + 4 - 4 > 13 - 4$

Simplify: $x > 9$



Solving Inequalities Using Multiplication and Division

We can also solve inequalities by multiplying or dividing both sides by a constant. For example, to solve the inequality $5x < 3$, we would divide both sides by 5 to get $x < \frac{3}{5}$.

However, something different happens when we multiply or divide by a negative number. We know, for example, that 5 is greater than 3. But if we multiply both sides of the inequality $5 > 3$ by -2 , we get $-10 > -6$. And we know that's not true; -10 is less than -6 .

This happens whenever we multiply or divide an inequality by a negative number, and so we have to flip the sign around to make the inequality true. For example, to multiply $2 < 4$ by -3 , first we multiply the 2 and the 4 each by -3 , and then we change the $<$ sign to a $>$ sign, so we end up with $-6 > -12$.

The same principle applies when the inequality contains variables.

Example 5

Solve each inequality.

a) $4x < 24$

b) $-5x \leq 21$

c) $\frac{x}{25} < \frac{3}{2}$

d) $\frac{x}{-7} \geq 9$

Solution

a) Original problem: $4x < 24$

Divide both sides by 4: $\frac{4x}{4} < \frac{24}{4}$

Simplify: $x < 6$

b) Original problem: $-5x \leq 21$

Divide both sides by -5 : $\frac{-5x}{-5} \geq \frac{21}{-5}$ *Flip the inequality sign.*

Simplify: $x \geq -\frac{21}{5}$

c) Original problem: $\frac{x}{25} < \frac{3}{2}$

Multiply both sides by 25: $25 \cdot \frac{x}{25} < \frac{3}{2} \cdot 25$

Simplify: $x < \frac{75}{2}$ or $x < 37.5$

d) Original problem: $\frac{x}{-7} \geq 9$

Multiply both sides by -7 : $-7 \cdot \frac{x}{-7} \leq 9 \cdot (-7)$ *Flip the inequality sign.*

Simplify: $x \leq -63$

Solving Multi-Step Inequalities

In the last two sections, we considered very simple inequalities which required one step to obtain the solution. However, most inequalities require several steps to arrive at the solution. As with solving equations, we must use the order of operations to find the correct solution. In addition, remember that **when we multiply or divide the inequality by a negative number, the direction of the inequality changes.**

The general procedure for solving multi-step inequalities is almost exactly like the procedure for solving multi-step equations:

1. Clear parentheses on both sides of the inequality and collect like terms.
2. Add or subtract terms so the variable is on one side and the constant is on the other side of the inequality sign.
3. Multiply and divide by whatever constants are attached to the variable. Remember to change the direction of the inequality if you multiply or divide by a negative number.

Example 6

Solve each of the following inequalities and graph the solution set.

a) $\frac{9x}{5} - 7 \geq -3x + 12$

b) $-25x + 12 \leq -10x - 12$

Solution

a) Original problem: $\frac{9x}{5} - 7 \geq -3x + 12$

Add $3x$ to both sides: $\frac{9x}{5} + 3x - 7 \geq -3x + 3x + 12$

Simplify: $\frac{24x}{5} - 7 \geq 12$

Add 7 to both sides: $\frac{24x}{5} - 7 + 7 \geq 12 + 7$

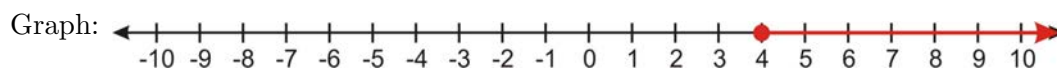
Simplify: $\frac{24x}{5} \geq 19$

Multiply 5 to both sides: $5 \cdot \frac{24x}{5} \geq 5 \cdot 19$

Simplify: $24x \geq 95$

Divide both sides by 24: $\frac{24x}{24} \geq \frac{95}{24}$

Simplify: $x \geq \frac{95}{24}$ **Answer**



b) Original problem: $-25x + 12 \leq -10x - 12$

Add $10x$ to both sides: $-25x + 10x + 12 \leq -10x + 10x - 12$

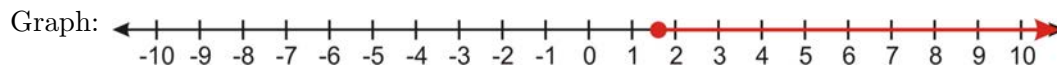
Simplify: $-15x + 12 \leq -12$

Subtract 12: $-15x + 12 - 12 \leq -12 - 12$

Simplify: $-15x \leq -24$

Divide both sides by -15 : $\frac{-15x}{-15} \geq \frac{-24}{-15}$ *flip the inequality sign*

Simplify: $x \geq \frac{8}{5}$ **Answer**



Example 7

Solve the following inequalities.

a) $4x - 2(3x - 9) \leq -4(2x - 9)$

b) $\frac{5x-1}{4} > -2(x + 5)$

Solution

a) Original problem: $4x - 2(3x - 9) \leq -4(2x - 9)$

Simplify parentheses: $4x - 6x + 18 \leq -8x + 36$

Collect like terms: $-2x + 18 \leq -8x + 36$

Add $8x$ to both sides: $-2x + 8x + 18 \leq -8x + 8x + 36$

Simplify: $6x + 18 \leq 36$

Subtract 18: $6x + 18 - 18 \leq 36 - 18$

Simplify: $6x \leq 18$

Divide both sides by 6: $\frac{6x}{6} \leq \frac{18}{6}$

Simplify: $x \leq 3$ **Answer**

b) Original problem: $\frac{5x-1}{4} > -2(x + 5)$

Simplify parenthesis: $\frac{5x-1}{4} > -2x - 10$

Multiply both sides by 4: $4 \cdot \frac{5x-1}{4} > 4(-2x - 10)$

Simplify: $5x - 1 > -8x - 40$

Add $8x$ to both sides: $5x + 8x - 1 > -8x + 8x - 40$

Simplify: $13x - 1 > -40$

Add 1 to both sides: $13x - 1 + 1 > -40 + 1$

Simplify: $13x > -39$

Divide both sides by 13: $\frac{13x}{13} > -\frac{39}{13}$

Simplify: $x > -3$ **Answer**

Further Practice

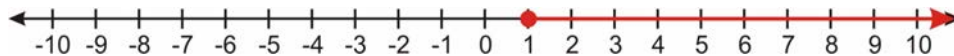
For additional practice solving inequalities, try the online game at <http://www.aaamath.com/equ725x7.htm#section2>. If you're having a hard time with multi-step inequalities, the video at <http://www.schooltube.com/video/aa66df49e0af4f85a5e9/MultiStep-Inequalities> will walk you through a few.

Lesson Summary

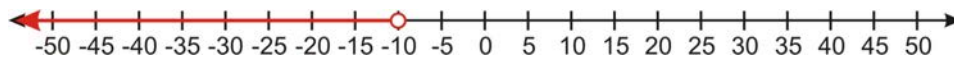
- The answer to an **inequality** is usually an **interval of values**.
- Solving inequalities works just like solving an equation. To solve, we isolate the variable on one side of the equation.
- When multiplying or dividing both sides of an inequality by a negative number, you need to **reverse the inequality**.

Review Questions

1. Write the inequality represented by the graph.



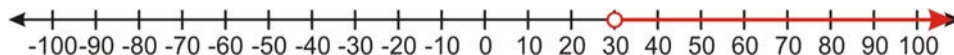
2. Write the inequality represented by the graph.



3. Write the inequality represented by the graph.



4. Write the inequality represented by the graph.



Graph each inequality on the number line.

5. $x < -35$
6. $x > -17$
7. $x \geq 20$
8. $x \leq 3$

Solve each inequality and graph the solution on the number line.

9. $x - 5 < 35$
10. $x + 15 \geq -60$
11. $x - 2 \leq 1$
12. $x - 8 > -20$
13. $x + 11 > 13$
14. $x + 65 < 100$

15. $x - 32 \leq 0$
16. $x + 68 \geq 75$

Solve each inequality. Write the solution as an inequality and graph it.

17. $3x \leq 6$
18. $\frac{x}{5} > -\frac{3}{10}$
19. $-10x > 250$
20. $\frac{x}{-7} \geq -5$

Solve each multi-step inequality.

21. $x - 5 > 2x + 3$
22. $2(x - 3) \leq 3x - 2$
23. $\frac{x}{3} < x + 7$
24. $\frac{3(x-4)}{12} \leq \frac{2x}{3}$
25. $2\left(\frac{x}{4} + 3\right) > 6(x - 1)$
26. $9x + 4 \leq -2\left(x + \frac{1}{2}\right)$

6.2 Using Inequalities

Learning Objectives

- Express answers to inequalities in a variety of ways.
- Identify the number of solutions of an inequality.
- Solve real-world problems using inequalities.

Introduction

Ms. Jerome wants to buy identical boxes of art supplies for her 25 students. If she can spend no more than \$375 on art supplies, what inequality describes the price can she afford for each individual box of supplies?

Expressing Solutions of an Inequality

The solution of an inequality can be expressed in four different ways:

1. **Inequality notation** The answer is simply expressed as $x < 15$.
2. **Set notation** The answer is expressed as a set: $\{x|x < 15\}$. The brackets indicate a set and the vertical line means “such that,” so we read this expression as “the set of all values of x such that x is a real number less than 15”.
3. **Interval notation** uses brackets to indicate the range of values in the solution. For example, the answer to our problem would be expressed as $(-\infty, 15)$, meaning “the interval containing all the numbers from $-\infty$ to 15 but not actually including $-\infty$ or 15”.
 - (a) Square or **closed brackets** “[” and “]” indicate that the number next to the bracket is included in the solution set.

- (b) Round or **open brackets** “(” and “)” indicate that the number next to the bracket is not included in the solution set. When using **infinity** and **negative infinity** (∞ and $-\infty$), we always use open brackets, because infinity isn’t an actual number and so it can’t ever really be included in an interval.

4. **Solution graph** shows the solution on the real number line. A closed circle on a number indicates that the number is included in the solution set, while an open circle indicates that the number is not included in the set. For our example, the solution graph is:



Example 1

- a) $[-4, 6]$ means that the solutions is all numbers between -4 and 6 **including** -4 and 6.
- b) $(8, 24)$ means that the solution is all numbers between 8 and 24 **not including** the numbers 8 and 24.
- c) $[3, 12)$ means that the solution is all numbers between 3 and 12, **including** 3 but **not including** 12.
- d) $(-10, \infty)$ means that the solution is all numbers greater than -10, **not including** -10.
- e) $(-\infty, \infty)$ means that the solution is all real numbers.

Identify the Number of Solutions of an Inequality

Inequalities can have:

- A set that has an infinite number of solutions.
- A set that has a discrete number of solutions.
- No solutions.

The inequalities we have solved so far all have an infinite number of solutions, at least in theory. For example, the inequality $\frac{5x-1}{4} > -2(x+5)$ has the solution $x > -3$. This solution says that all real numbers greater than -3 make this inequality true, and there are infinitely many such numbers.

However, in real life, sometimes we are trying to solve a problem that can only have positive integer answers, because the answers describe numbers of discrete objects.

For example, suppose you are trying to figure out how many \$8 CDs you can buy if you want to spend less than \$50. An inequality to describe this situation would be $8x < 50$, and if you solved that inequality you would get $x < \frac{50}{8}$, or $x < 6.25$.

But could you really buy *any* number of CDs as long as it’s less than 6.25? No; you couldn’t really buy 6.1 CDs, or -5 CDs, or any other fractional or negative number of CDs. So if we wanted to express our solution in set notation, we couldn’t express it as the set of all numbers less than 6.25, or $\{x|x < 6.25\}$. Instead, the solution is just the set containing all the *nonnegative whole numbers* less than 6.25, or $\{0, 1, 2, 3, 4, 5, 6\}$. When we’re solving a real-world problem dealing with discrete objects like CDs, our solution set will often be a finite set of numbers instead of an infinite interval.

An inequality can also have no solutions at all. For example, consider the inequality $x - 5 > x + 6$. When we subtract x from both sides, we end up with $-5 > 6$, which is not true for any value of x . We say that this inequality has no solution.

The opposite can also be true. If we flip the inequality sign in the above inequality, we get $x - 5 < x + 6$, which simplifies to $-5 < 6$. That’s always true no matter what x is, so the solution to that inequality would be all real numbers, or $(-\infty, \infty)$.

Solve Real-World Problems Using Inequalities

Solving real-world problems that involve inequalities is very much like solving problems that involve equations.

Example 2

In order to get a bonus this month, Leon must sell at least 120 newspaper subscriptions. He sold 85 subscriptions in the first three weeks of the month. How many subscriptions must Leon sell in the last week of the month?

Solution

Let x = the number of subscriptions Leon sells in the last week of the month. The total number of subscriptions for the month must be greater than 120, so we write $85 + x \geq 120$. We solve the inequality by subtracting 85 from both sides: $x \geq 35$.

Leon must sell 35 or more subscriptions in the last week to get his bonus.

To check the answer, we see that $85 + 35 = 120$. If he sells 35 or more subscriptions, the total number of subscriptions he sells that month will be 120 or more. **The answer checks out.**

Example 3

Virena's Scout troop is trying to raise at least \$650 this spring. How many boxes of cookies must they sell at \$4.50 per box in order to reach their goal?

Solution

Let x = number of boxes sold. Then the inequality describing this problem is $4.50x \geq 650$.

We solve the inequality by dividing both sides by 4.50: $x \geq 144.44$.

We round up the answer to 145 since only whole boxes can be sold.

Virena's troop must sell at least 145 boxes.

If we multiply 145 by \$4.50 we obtain \$652.50, so if Virena's troop sells more than 145 boxes they will raise more than \$650. But if they sell 144 boxes, they will only raise \$648, which is not enough. So they must indeed sell at least 145 boxes. **The answer checks out.**

Example 4

The width of a rectangle is 20 inches. What must the length be if the perimeter is at least 180 inches?

Solution

Let x = length of the rectangle. The formula for perimeter is

$$\text{Perimeter} = 2 \times \text{length} + 2 \times \text{width}$$

Since the perimeter must be at least 180 inches, we have $2x + 2(20) \geq 180$.

Simplify: $2x + 40 \geq 180$

Subtract 40 from both sides: $2x \geq 140$

Divide both sides by 2: $x \geq 70$

The length must be at least 70 inches.

If the length is at least 70 inches and the width is 20 inches, then the perimeter is at least $2(70) + 2(20) = 180$ inches. **The answer checks out.**

Further Practice

The videos at <http://www.youtube.com/watch?v=k9JSbMfFZ9U&feature=related> and <http://www.youtube.com/watch?v=ArzPkaqym50> contain more examples of real-world problems using inequalities.

Lesson Summary

- Inequalities can have infinite solutions, no solutions, or discrete solutions.
- There are four ways to represent an inequality: *Equation notation*, *set notation*, *interval notation*, and *solution graph*.

Review Questions

Solve each inequality. Give the solution in inequality notation and interval notation.

1. $x + 15 < 12$
2. $x - 4 \geq 13$
3. $9x > -\frac{3}{4}$
4. $-\frac{x}{15} \leq 5$
5. $620x > 2400$
6. $\frac{x}{20} \geq -\frac{7}{40}$
7. $\frac{3x}{5} > \frac{3}{5}$
8. $x + 3 > x - 2$

Solve each inequality. Give the solution in inequality notation and set notation.

9. $x + 17 < 3$
10. $x - 12 \geq 80$
11. $-0.5x \leq 7.5$
12. $75x \geq 125$
13. $\frac{x}{-3} > -\frac{10}{9}$
14. $\frac{x}{-15} < 8$
15. $\frac{x}{4} > \frac{5}{4}$
16. $3x - 7 \geq 3(x - 7)$

Solve the following inequalities, give the solution in set notation, and show the solution graph.

17. $4x + 3 < -1$
18. $2x < 7x - 36$
19. $5x > 8x + 27$
20. $5 - x < 9 + x$
21. $4 - 6x \leq 2(2x + 3)$
22. $5(4x + 3) \geq 9(x - 2) - x$
23. $2(2x - 1) + 3 < 5(x + 3) - 2x$
24. $8x - 5(4x + 1) \geq -1 + 2(4x - 3)$
25. $9 \cdot 2(7x - 2) - 3(x + 2) < 4x - (3x + 4)$
26. $\frac{2}{3}x - \frac{1}{2}(4x - 1) \geq x + 2(x - 3)$

27. At the San Diego Zoo you can either pay \$22.75 for the entrance fee or \$71 for the yearly pass which entitles you to unlimited admission.
- At most how many times can you enter the zoo for the \$22.75 entrance fee before spending more than the cost of a yearly membership?
 - Are there infinitely many or finitely many solutions to this inequality?
28. Proteek's scores for four tests were 82, 95, 86, and 88. What will he have to score on his fifth and last test to average at least 90 for the term?

6.3 Compound Inequalities

Learning Objectives

- Write and graph compound inequalities on a number line.
- Solve compound inequalities with “*and*.”
- Solve compound inequalities with “*or*.”
- Solve compound inequalities using a graphing calculator (TI family).
- Solve real-world problems using compound inequalities.

Introduction

In this section, we'll solve compound inequalities—inequalities with more than one constraint on the possible values the solution can have.

There are two types of compound inequalities:

- Inequalities joined by the word “*and*,” where the solution is a set of values greater than a number *and* less than another number. We can write these inequalities in the form “ $x > a$ and $x < b$,” but usually we just write “ $a < x < b$.” Possible values for x are ones that will make *both* inequalities true.
- Inequalities joined by the word “*or*,” where the solution is a set of values greater than a number *or* less than another number. We write these inequalities in the form “ $x > a$ or $x < b$.” Possible values for x are ones that will make *at least one* of the inequalities true.

You might wonder why the variable x has to be *greater than* one number and/or *less than* the other number; why can't it be greater than both numbers, or less than both numbers? To see why, let's take an example.

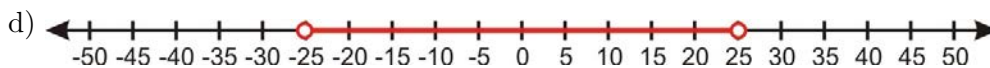
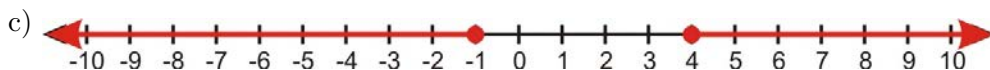
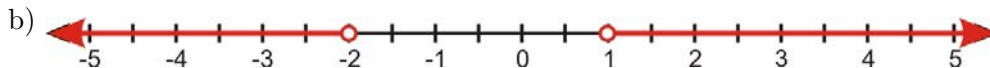
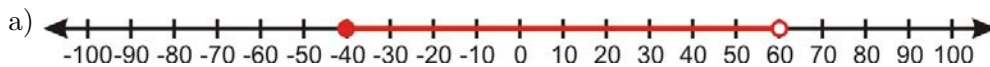
Consider the compound inequality “ $x > 5$ and $x > 3$.” Are there any numbers greater than 5 that are *not* greater than 3? No! Since 5 is greater than 3, everything greater than 5 is also greater than 3. If we say x is greater than both 5 and 3, that doesn't tell us any more than if we just said x is greater than 5. So this compound inequality isn't really compound; it's equivalent to the simple inequality $x > 5$. And that's what would happen no matter which two numbers we used; saying that x is greater than both numbers is just the same as saying that x is greater than the bigger number, and saying that x is less than both numbers is just the same as saying that x is less than the smaller number.

Compound inequalities with “*or*” work much the same way. Every number that's greater than 3 *or* greater than 5 is also just plain greater than 3, and every number that's greater than 3 is certainly greater than 3 *or* greater than 5—so if we say “ $x > 5$ or $x > 3$,” that's the same as saying just “ $x > 3$.” Saying that x is greater than at least one of two numbers is just the same as saying that x is greater than the smaller number, and saying that x is less than at least one of two numbers is just the same as saying that x is less than the greater number.

Write and Graph Compound Inequalities on a Number Line

Example 1

Write the inequalities represented by the following number line graphs.



Solution

a) The solution graph shows that the solution is any value between -40 and 60 , including -40 but not 60 . Any value in the solution set satisfies both $x \geq -40$ and $x < 60$.

This is usually written as $-40 \leq x < 60$.

b) The solution graph shows that the solution is any value greater than 1 (not including 1) or any value less than -2 (not including -2). You can see that there can be no values that can satisfy both these conditions at the same time. We write: $x > 1$ or $x < -2$.

c) The solution graph shows that the solution is any value greater than 4 (including 4) or any value less than -1 (including -1). We write: $x \geq 4$ or $x \leq -1$.

d) The solution graph shows that the solution is any value that is both less than 25 (not including 25) and greater than -25 (not including -25). Any value in the solution set satisfies both $x > -25$ and $x < 25$.

This is usually written as $-25 < x < 25$.

Example 2

Graph the following compound inequalities on a number line.

a) $-4 \leq x \leq 6$

b) $x < 0$ or $x > 2$

c) $x \geq -8$ or $x \leq -20$

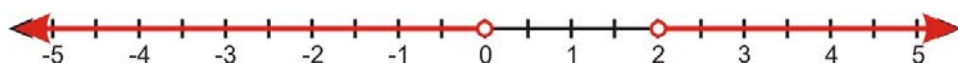
d) $-15 < x \leq 85$

Solution

a) The solution is all numbers between -4 and 6 , including both -4 and 6 .



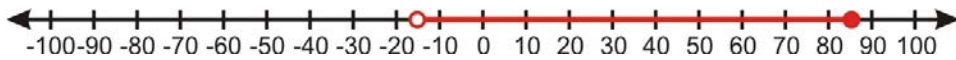
b) The solution is all numbers less than 0 or greater than 2 , not including 0 or 2 .



c) The solution is all numbers greater than or equal to -8 or less than or equal to -20 .



d) The solution is all numbers between -15 and 85, not including -15 but including 85.



Solve a Compound Inequality With “and” or “or”

When we solve compound inequalities, we separate the inequalities and solve each of them separately. Then, we combine the solutions at the end.

Example 3

Solve the following compound inequalities and graph the solution set.

a) $-2 < 4x - 5 \leq 11$

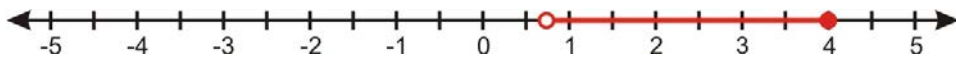
b) $3x - 5 < x + 9 \leq 5x + 13$

Solution

a) First we re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{rcl} -2 < 4x - 5 & & 4x - 5 \leq 11 \\ 3 < 4x & \text{and} & 4x \leq 16 \\ \frac{3}{4} < x & & x \leq 4 \end{array}$$

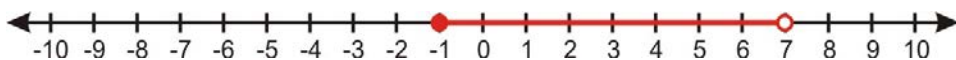
Answer: $\frac{3}{4} < x$ and $x \leq 4$. This can be written as $\frac{3}{4} < x \leq 4$.



b) Re-write the compound inequality as two separate inequalities with *and*. Then solve each inequality separately.

$$\begin{array}{rcl} 3x - 5 < x + 9 & & x + 9 \leq 5x + 13 \\ 2x < 14 & \text{and} & -4 \leq 4x \\ x < 7 & & -1 \leq x \end{array}$$

Answer: $x < 7$ and $x \geq -1$. This can be written as: $-1 \leq x < 7$.



Example 4

Solve the following compound inequalities and graph the solution set.

a) $9 - 2x \leq 3$ or $3x + 10 \leq 6 - x$

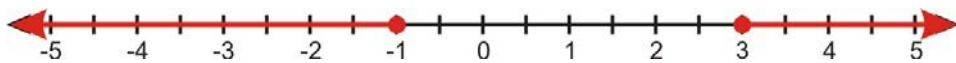
b) $\frac{x-2}{6} \leq 2x - 4$ or $\frac{x-2}{6} > x + 5$

Solution

a) Solve each inequality separately:

$$\begin{array}{rcl} 9 - 2x \leq 3 & & 3x + 10 \leq 6 - x \\ -2x \leq -6 & \text{or} & 4x \leq -4 \\ x \geq 3 & & x \leq -1 \end{array}$$

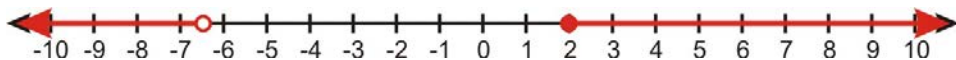
Answer: $x \geq 3$ or $x \leq -1$



b) Solve each inequality separately:

$$\begin{array}{l} \frac{x-2}{6} \leq 2x-4 \\ x-2 \leq 6(2x-4) \\ x-2 \leq 12x-24 \quad \text{or} \quad x-2 > 6(x+5) \\ 22 \leq 11x \\ 2 \leq x \end{array} \quad \begin{array}{l} \frac{x-2}{6} > x+5 \\ x-2 > 6(x+5) \\ x-2 > 6x+30 \\ -32 > 5x \\ -6.4 > x \end{array}$$

Answer: $x \geq 2$ or $x < -6.4$



The video at <http://www.math-videos-online.com/solve-compound-inequality.html> shows the process of solving and graphing compound inequalities in more detail. One thing you may notice in this video is that in the second problem, the two solutions joined with “or” overlap, and so the solution ends up being the set of all real numbers, or $(-\infty, \infty)$. This happens sometimes with compound inequalities that involve “or”; for example, if the solution to an inequality ended up being “ $x < 5$ or $x > 1$,” the solution set would be all real numbers. This makes sense if you think about it: all real numbers are either a) less than 5, or b) greater than or equal to 5, and the ones that are greater than or equal to 5 are also greater than 1—so all real numbers are either a) less than 5 or b) greater than 1.

Compound inequalities with “and,” meanwhile, can turn out to have *no* solutions. For example, the inequality “ $x < 3$ and $x > 4$ ” has no solutions: no number is both greater than 4 and less than 3. If we write it as $4 < x < 3$ it’s even more obvious that it has no solutions; $4 < x < 3$ implies that $4 < 3$, which is false.

Solve Compound Inequalities Using a Graphing Calculator (TI-83/84 family)

Graphing calculators can show you the solution to an inequality in the form of a graph. This can be especially useful when dealing with compound inequalities.

Example 5

Solve the following inequalities using a graphing calculator.

a) $5x + 2(x - 3) \geq 2$

b) $7x - 2 < 10x + 1 < 9x + 5$

c) $3x + 2 \leq 10$ or $3x + 2 \geq 15$

Solution

a) Press the [Y=] button and enter the inequality on the first line of the screen.

```

Plot1 Plot2 Plot3
\Y1=5X+2(X-3)≥2
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=

```

(To get the \geq symbol, press [TEST] [2nd] [MATH] and choose option 4.)

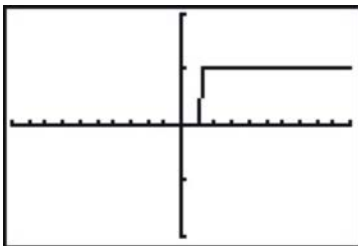
```

[TEST] LOGIC
1: =
2: ≠
3: <
4: ≥
5: <
6: ≤

```

Then press the [GRAPH] button.

Because the calculator uses the number 1 to mean “true” and 0 to mean “false,” you will see a step function with the y -value jumping from 0 to 1.



The solution set is the values of x for which the graph shows $y = 1$ —in other words, the set of x -values that make the inequality true.

X	Y1
1.13	0
1.14	0
1.15	1
1.16	1
1.17	1
1.18	1
1.19	1

X=1.14

Note: You may need to press the [WINDOW] key or the [ZOOM] key to adjust the window to see the full graph.

The solution is $x > \frac{8}{7}$, which is why you can see the y -value changing from 0 to 1 at about 1.14.

b) This is a compound inequality: $7x - 2 < 10x + 1$ and $10x + 1 < 9x + 5$. You enter it like this:

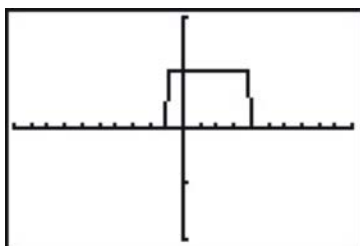
```

Plot1 Plot2 Plot3
\Y1=(7X-2<10X+1)
and (10X+1<9X+5)
)
\Y2=
\Y3=
\Y4=
\Y5=

```

(To find the [AND] symbol, press [TEST], choose [LOGIC] on the top row and choose option 1.)

The resulting graph should look like this:



The solution are the values of x for which $y = 1$; in this case that would be $-1 < x < 4$.

c) This is another compound inequality.

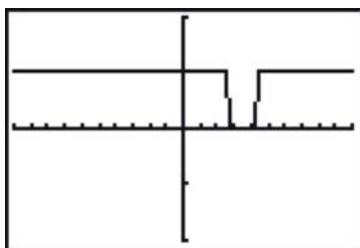
```

Plot1 Plot2 Plot3
\Y1=(3X+2≤10) or
(3X+2≥15)
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=

```

(To enter the [OR] symbol, press [TEST], choose [LOGIC] on the top row and choose option 2.)

The resulting graph should look like this:



The solution are the values of x for which $y = 1$ —in this case, $x \leq 2.7$ or $x \geq 4.3$.

Solve Real-World Problems Using Compound Inequalities

Many application problems require the use of compound inequalities to find the solution.

Example 6

The speed of a golf ball in the air is given by the formula $v = -32t + 80$. When is the ball traveling between 20 ft/sec and 30 ft/sec?

Solution

First we set up the inequality $20 \leq v \leq 30$, and then replace v with the formula $v = -32t + 80$ to get $20 \leq -32t + 80 \leq 30$.

Then we separate the compound inequality and solve each separate inequality:

$$\begin{array}{rcl} 20 \leq -32t + 80 & & -32t + 80 \leq 30 \\ 32t \leq 60 & \text{and} & 50 \leq 32t \\ t \leq 1.875 & & 1.56 \leq t \end{array}$$

Answer: $1.56 \leq t \leq 1.875$

To check the answer, we plug in the minimum and maximum values of t into the formula for the speed.

For $t = 1.56$, $v = -32t + 80 = -32(1.56) + 80 = 30$ ft/sec

For $t = 1.875$, $v = -32t + 80 = -32(1.875) + 80 = 20$ ft/sec

So the speed is between 20 and 30 ft/sec. **The answer checks out.**

Example 7

William's pick-up truck gets between 18 to 22 miles per gallon of gasoline. His gas tank can hold 15 gallons of gasoline. If he drives at an average speed of 40 miles per hour, how much driving time does he get on a full tank of gas?

Solution

Let t = driving time. We can use dimensional analysis to get from time per tank to miles per gallon:

$$\frac{t \text{ hours}}{1 \text{ tank}} \times \frac{1 \text{ tank}}{15 \text{ gallons}} \times \frac{40 \text{ miles}}{1 \text{ hour}} \times \frac{40t \text{ miles}}{15 \text{ gallon}}$$

Since the truck gets between 18 and 22 miles/gallon, we set up the compound inequality $18 \leq \frac{40t}{15} \leq 22$. Then we separate the compound inequality and solve each inequality separately:

$$\begin{array}{rcl} 18 \leq \frac{40t}{15} & & \frac{40t}{15} \leq 22 \\ 270 \leq 40t & \text{and} & 40t \leq 330 \\ 6.75 \leq t & & t \leq 8.25 \end{array}$$

Answer: $6.75 \leq t \leq 8.25$.

Andrew can drive between 6.75 and 8.25 hours on a full tank of gas.

If we plug in $t = 6.75$ we get $\frac{40t}{15} = \frac{40(6.75)}{15} = 18$ miles per gallon.

If we plug in $t = 8.25$ we get $\frac{40t}{15} = \frac{40(8.25)}{15} = 22$ miles per gallon.

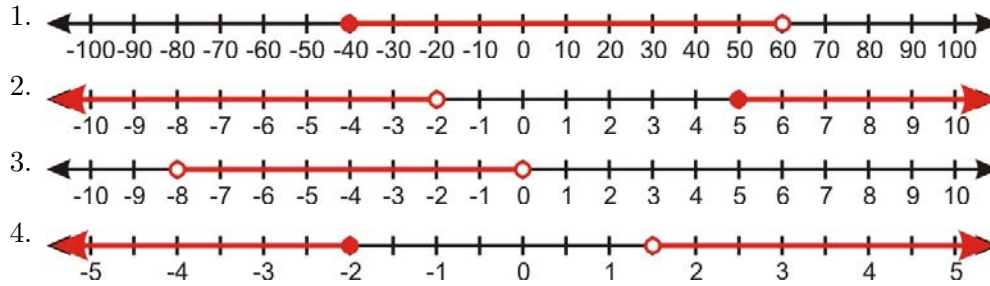
The answer checks out.

Lesson Summary

- **Compound inequalities** combine two or more inequalities with “**and**” or “**or**.”
- “**And**” combinations mean that only solutions for *both* inequalities will be solutions to the compound inequality.
- “**Or**” combinations mean solutions to *either* inequality will also be solutions to the compound inequality.

Review Questions

Write the compound inequalities represented by the following graphs.



Solve the following compound inequalities and graph the solution on a number line.

- $-5 \leq x - 4 \leq 13$
- $1 \leq 3x + 5 \leq 4$
- $-12 \leq 2 - 5x \leq 7$
- $\frac{3}{4} \leq 2x + 9 \leq \frac{3}{2}$
- $-2 \leq \frac{2x-1}{3} < -1$
- $4x - 1 \geq 7$ or $\frac{9x}{2} < 3$
- $3 - x < -4$ or $3 - x > 10$
- $\frac{2x+3}{4} < 2$ or $-\frac{x}{5} + 3 < \frac{2}{5}$
- $2x - 7 \leq -3$ or $2x - 3 > 11$
- $4x + 3 < 9$ or $-5x + 4 \leq -12$
- How would you express the answer to problem 5 as a set?
- How would you express the answer to problem 5 as an interval?
- How would you express the answer to problem 10 as a set?
- (a) Could you express the answer to problem 10 as a single interval? Why or why not?
(b) How would you express the first part of the solution in interval form?
(c) How would you express the second part of the solution in interval form?
- Express the answers to problems 1 and 3 in interval notation.
- Express the answers to problems 6 through 9 in interval notation.
- Solve the inequality " $x \geq -3$ or $x < 1$ " and express the answer in interval notation.
- How many solutions does the inequality " $x \geq 2$ and $x \leq 2$ " have?
- To get a grade of B in her Algebra class, Stacey must have an average grade greater than or equal to 80 and less than 90. She received the grades of 92, 78, 85 on her first three tests.
 - Between which scores must her grade on the final test fall if she is to receive a grade of B for the class? (Assume all four tests are weighted the same.)
 - What range of scores on the final test would give her an overall grade of C, if a C grade requires an average score greater than or equal to 70 and less than 80?
 - If an A grade requires a score of at least 90, and the maximum score on a single test is 100, is it possible for her to get an A in this class? (Hint: look again at your answer to part a.)

6.4 Absolute Value Equations and Inequalities

Learning Objectives

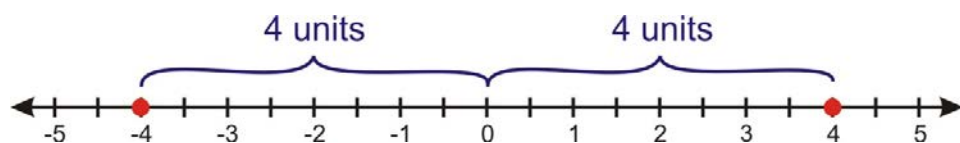
- Solve an absolute value equation.

- Analyze solutions to absolute value equations.
- Graph absolute value functions.
- Solve absolute value inequalities.
- Rewrite and solve absolute value inequalities as compound inequalities.
- Solve real-world problems using absolute value equations and inequalities.

Introduction

Timmy is trying out his new roller skates. He's not allowed to cross the street yet, so he skates back and forth in front of his house. If he skates 20 yards east and then 10 yards west, how far is he from where he started? What if he skates 20 yards west and then 10 yards east?

The **absolute value** of a number is its distance from zero on a number line. There are always two numbers on the number line that are the same distance from zero. For instance, the numbers 4 and -4 are each a distance of 4 units away from zero.



$|4|$ represents the distance from 4 to zero, which equals 4.

$|-4|$ represents the distance from -4 to zero, which also equals 4.

In fact, for any real number x :

$|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.

Absolute value has no effect on a positive number, but changes a negative number into its positive inverse.

Example 1

Evaluate the following absolute values.

a) $|25|$

b) $|-120|$

c) $|-3|$

d) $|55|$

e) $|\frac{-5}{4}|$

Solution

a) $|25| = 25$ Since 25 is a positive number, the absolute value does not change it.

b) $|-120| = 120$ Since -120 is a negative number, the absolute value makes it positive.

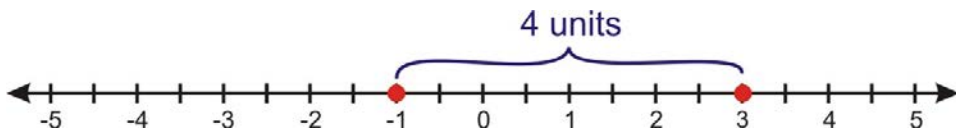
c) $|-3| = 3$ Since -3 is a negative number, the absolute value makes it positive.

d) $|55| = 55$ Since 55 is a positive number, the absolute value does not change it.

e) $|\frac{-5}{4}| = \frac{5}{4}$ Since $\frac{-5}{4}$ is a negative number, the absolute value makes it positive.

Absolute value is very useful in finding the distance between two points on the number line. The **distance** between any two points a and b on the number line is $|a - b|$ or $|b - a|$.

For example, the distance from 3 to -1 on the number line is $|3 - (-1)| = |4| = 4$.



We could have also found the distance by subtracting in the opposite order: $|-1 - 3| = |-4| = 4$. This makes sense because the distance is the same whether you are going from 3 to -1 or from -1 to 3.

Example 2

Find the distance between the following points on the number line.

- a) 6 and 15
- b) -5 and 8
- c) -3 and -12

Solution

Distance is the absolute value of the difference between the two points.

- a) distance = $|6 - 15| = |-9| = 9$
- b) distance = $|-5 - 8| = |-13| = 13$
- c) distance = $|-3 - (-12)| = |9| = 9$

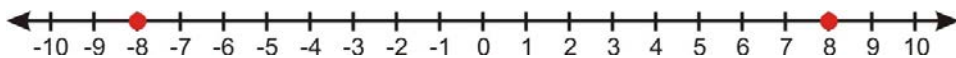
Remember: When we computed the change in x and the change in y as part of the slope computation, these values were positive or negative, depending on the direction of movement. In this discussion, “distance” means a positive distance only.

Solve an Absolute Value Equation

We now want to solve equations involving absolute values. Consider the following equation:

$$|x| = 8$$

This means that the distance from the number x to zero is 8. There are two numbers that satisfy this condition: 8 and -8.



When we solve absolute value equations we always consider two possibilities:

1. The expression inside the absolute value sign is not negative.
2. The expression inside the absolute value sign is negative.

Then we solve each equation separately.

Example 3

Solve the following absolute value equations.

- a) $|x| = 3$
- b) $|x| = 10$

Solution

- a) There are two possibilities: $x = 3$ **and** $x = -3$.
- b) There are two possibilities: $x = 10$ **and** $x = -10$.

Analyze Solutions to Absolute Value Equations

Example 4

Solve the equation $|x - 4| = 5$ and interpret the answers.

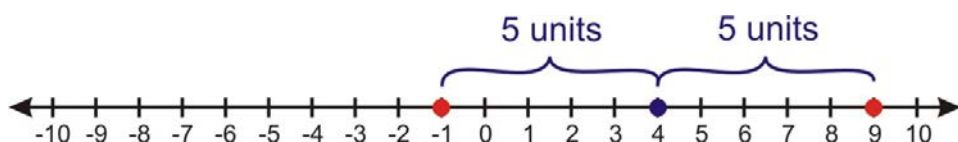
Solution

We consider two possibilities: the expression inside the absolute value sign is nonnegative or is negative. Then we solve each equation separately.

$$\begin{aligned}x - 4 &= 5 & \text{and} & & x - 4 &= -5 \\x &= 9 & & & x &= -1\end{aligned}$$

$x = 9$ and $x = -1$ are the solutions.

The equation $|x - 4| = 5$ can be interpreted as “what numbers on the number line are 5 units away from the number 4?” If we draw the number line we see that there are two possibilities: 9 and -1.



Example 5

Solve the equation $|x + 3| = 2$ and interpret the answers.

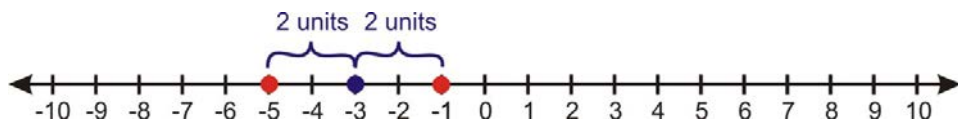
Solution

Solve the two equations:

$$\begin{aligned}x + 3 &= 2 & \text{and} & & x + 3 &= -2 \\x &= -1 & & & x &= -5\end{aligned}$$

$x = -5$ and $x = -1$ are the answers.

The equation $|x + 3| = 2$ can be re-written as: $|x - (-3)| = 2$. We can interpret this as “what numbers on the number line are 2 units away from -3?” There are two possibilities: -5 and -1.



Example 6

Solve the equation $|2x - 7| = 6$ and interpret the answers.

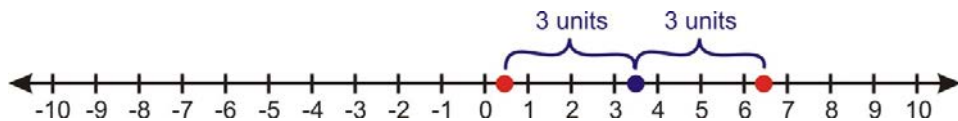
Solution

Solve the two equations:

$$\begin{aligned}2x - 7 &= 6 & & & 2x - 7 &= -6 \\2x &= 13 & \text{and} & & 2x &= 1 \\x &= \frac{13}{2} & & & x &= \frac{1}{2}\end{aligned}$$

Answer: $x = \frac{13}{2}$ and $x = \frac{1}{2}$.

The interpretation of this problem is clearer if the equation $|2x - 7| = 6$ is divided by 2 on both sides to get $\frac{1}{2}|2x - 7| = 3$. Because $\frac{1}{2}$ is nonnegative, we can distribute it over the absolute value sign to get $|x - \frac{7}{2}| = 3$. The question then becomes “What numbers on the number line are 3 units away from $\frac{7}{2}$?” There are two answers: $\frac{13}{2}$ and $\frac{1}{2}$.



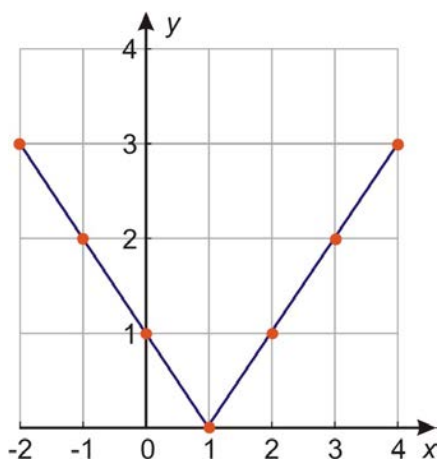
Graph Absolute Value Functions

Now let's look at how to graph absolute value functions.

Consider the function $y = |x - 1|$. We can graph this function by making a table of values:

Table 6.1:

x	$y = x - 1 $
-2	$y = -2 - 1 = -3 = 3$
-1	$y = -1 - 1 = -2 = 2$
0	$y = 0 - 1 = -1 = 1$
1	$y = 1 - 1 = 0 = 0$
2	$y = 2 - 1 = 1 = 1$
3	$y = 3 - 1 = 2 = 2$
4	$y = 4 - 1 = 3 = 3$



You can see that the graph of an absolute value function makes a big “V”. It consists of two line rays (or line segments), one with positive slope and one with negative slope, joined at the **vertex** or **cusp**.

We've already seen that to solve an absolute value equation we need to consider two options:

1. The expression inside the absolute value is not negative.
2. The expression inside the absolute value is negative.

Combining these two options gives us the two parts of the graph.

For instance, in the above example, the expression inside the absolute value sign is $x - 1$. By definition, this expression is nonnegative when $x - 1 \geq 0$, which is to say when $x \geq 1$. When the expression inside the absolute value sign is nonnegative, we can just drop the absolute value sign. So for all values of x greater than or equal to 1, the equation is just $y = x - 1$.

On the other hand, when $x - 1 < 0$ — in other words, when $x < 1$ — the expression inside the absolute value sign is negative. That means we have to drop the absolute value sign but also multiply the expression by -1 . So for all values of x less than 1, the equation is $y = -(x - 1)$, or $y = -x + 1$.

These are both graphs of straight lines, as shown above. They meet at the point where $x - 1 = 0$ — that is, at $x = 1$.

We can graph absolute value functions by breaking them down algebraically as we just did, or we can graph them using a table of values. However, when the absolute value equation is linear, the easiest way to graph it is to combine those two techniques, as follows:

1. Find the vertex of the graph by setting the expression inside the absolute value equal to zero and solving for x .
2. Make a table of values that includes the vertex, a value smaller than the vertex, and a value larger than the vertex. Calculate the corresponding values of y using the equation of the function.
3. Plot the points and connect them with two straight lines that meet at the vertex.

Example 7

Graph the absolute value function $y = |x + 5|$.

Solution

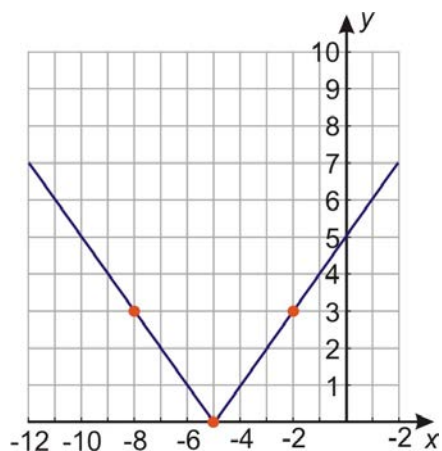
Step 1: Find the vertex by solving $x + 5 = 0$. The vertex is at $x = -5$.

Step 2: Make a table of values:

Table 6.2:

x	$y = x + 5 $
-8	$y = -8 + 5 = -3 = 3$
-5	$y = -5 + 5 = 0 = 0$
-2	$y = -2 + 5 = 3 = 3$

Step 3: Plot the points and draw two straight lines that meet at the vertex:



Example 8

Graph the absolute value function: $y = |3x - 12|$

Solution

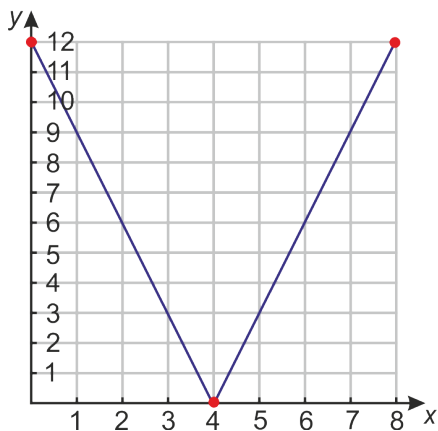
Step 1: Find the vertex by solving $3x - 12 = 0$. The vertex is at $x = 4$.

Step 2: Make a table of values:

Table 6.3:

x	$y = 3x - 12 $
0	$y = 3(0) - 12 = -12 = 12$
4	$y = 3(4) - 12 = 0 = 0$
8	$y = 3(8) - 12 = 12 = 12$

Step 3: Plot the points and draw two straight lines that meet at the vertex.



Solve Real-World Problems Using Absolute Value Equations

Example 9

A company packs coffee beans in airtight bags. Each bag should weigh 16 ounces, but it is hard to fill each bag to the exact weight. After being filled, each bag is weighed; if it is more than 0.25 ounces overweight or underweight, it is emptied and repacked. What are the lightest and heaviest acceptable bags?

Solution

The weight of each bag is allowed to be 0.25 ounces away from 16 ounces; in other words, the *difference* between the bag's weight and 16 ounces is allowed to be 0.25 ounces. So if x is the weight of a bag in ounces, then the equation that describes this problem is $|x - 16| = 0.25$.

Now we must consider the positive and negative options and solve each equation separately:

$$\begin{aligned}x - 16 &= 0.25 & \text{and} & & x - 16 &= -0.25 \\x &= 16.25 & & & x &= 15.75\end{aligned}$$

The lightest acceptable bag weighs 15.75 ounces and the heaviest weighs 16.25 ounces.

We see that $16.25 - 16 = 0.25$ ounces and $16 - 15.75 = 0.25$ ounces. The answers are 0.25 ounces bigger and smaller than 16 ounces respectively.

The answer checks out.

The answer you just found describes the lightest and heaviest acceptable bags of coffee beans. But how do we describe the total possible range of acceptable weights? That's where inequalities become useful once again.

Absolute Value Inequalities

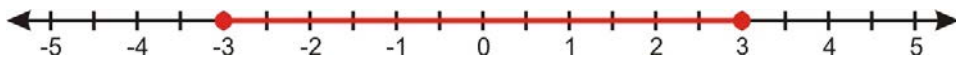
Absolute value inequalities are solved in a similar way to absolute value equations. In both cases, you must consider the same two options:

1. The expression inside the absolute value is not negative.
2. The expression inside the absolute value is negative.

Then you must solve each inequality separately.

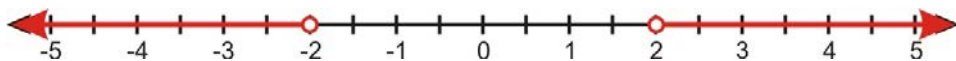
Solve Absolute Value Inequalities

Consider the inequality $|x| \leq 3$. Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero is less than or equal to 3. The following graph shows this solution:



Notice that this is also the graph for the compound inequality $-3 \leq x \leq 3$.

Now consider the inequality $|x| > 2$. Since the absolute value of x represents the distance from zero, the solutions to this inequality are those numbers whose distance from zero are more than 2. The following graph shows this solution.



Notice that this is also the graph for the compound inequality $x < -2$ or $x > 2$.

Example 1

Solve the following inequalities and show the solution graph.

- a) $|x| < 5$
- b) $|x| \geq 2.5$

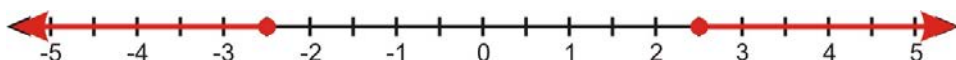
Solution

a) $|x| < 5$ represents all numbers whose distance from zero is less than 5.



This answer can be written as " $-5 < x < 5$ ".

b) $|x| \geq 2.5$ represents all numbers whose distance from zero is more than or equal to 2.5



This answer can be written as " $x \leq -2.5$ or $x \geq 2.5$ ".

Rewrite and Solve Absolute Value Inequalities as Compound Inequalities

In the last section you saw that absolute value inequalities are compound inequalities.

Inequalities of the type $|x| < a$ can be rewritten as “ $-a < x < a$ ”.

Inequalities of the type $|x| > b$ can be rewritten as “ $x < -b$ or $x > b$ ”.

To solve an absolute value inequality, we separate the expression into two inequalities and solve each of them individually.

Example 2

Solve the inequality $|x - 3| < 7$ and show the solution graph.


Solution

Re-write as a compound inequality: $-7 < x - 3 < 7$

Write as two separate inequalities: $x - 3 < 7$ and $x - 3 > -7$

Solve each inequality: $x < 10$ and $x > -4$

Re-write solution: $-4 < x < 10$

The solution graph is 

We can think of the question being asked here as “What numbers are within 7 units of 3?”; the answer can then be expressed as “All the numbers between -4 and 10.”

Example 3

Solve the inequality $|4x + 5| \leq 13$ and show the solution graph.

Solution

Re-write as a compound inequality: $-13 \leq 4x + 5 \leq 13$

Write as two separate inequalities: $4x + 5 \leq 13$ and $4x + 5 \geq -13$

Solve each inequality: $4x \leq 8$ and $4x \geq -18$

$x \leq 2$ and $x \geq -\frac{9}{2}$

Re-write solution: $-\frac{9}{2} \leq x \leq 2$

The solution graph is 


Example 4

Solve the inequality $|x + 12| > 2$ and show the solution graph.

Solution

Re-write as a compound inequality: $x + 12 < -2$ or $x + 12 > 2$

Solve each inequality: $x < -14$ or $x > -10$

The solution graph is 

Example 5

Solve the inequality $|8x - 15| \geq 9$ and show the solution graph.

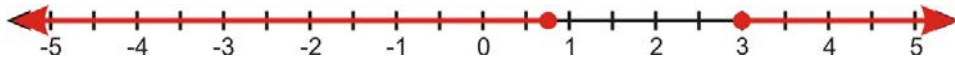
Solution

Re-write as a compound inequality: $8x - 15 \leq -9$ or $8x - 15 \geq 9$

Solve each inequality: $8x \leq 6$ or $8x \geq 24$

$x \leq \frac{3}{4}$ or $x \geq 3$

The solution graph is



Solve Real-World Problems Using Absolute Value Inequalities

Absolute value inequalities are useful in problems where we are dealing with a range of values.

Example 6

The velocity of an object is given by the formula $v = 25t - 80$, where the time is expressed in seconds and the velocity is expressed in feet per second. Find the times for which the magnitude of the velocity is greater than or equal to 60 feet per second.

Solution

The *magnitude* of the velocity is the absolute value of the velocity. If the velocity is $25t - 80$ feet per second, then its magnitude is $|25t - 80|$ feet per second. We want to find out when that magnitude is greater than or equal to 60, so we need to solve $|25t - 80| \geq 60$ for t .

First we have to split it up: $25t - 80 \geq 60$ or $25t - 80 \leq -60$

Then solve: $25t \geq 140$ or $25t \leq 20$

$t \geq 5.6$ or $t \leq 0.8$

The magnitude of the velocity is greater than 60 ft/sec for times **less than 0.8 seconds** and for times **greater than 5.6 seconds**.

When $t = 0.8$ seconds, $v = 25(0.8) - 80 = -60$ ft/sec. The magnitude of the velocity is 60 ft/sec. (The negative sign in the answer means that the object is moving backwards.)

When $t = 5.6$ seconds, $v = 25(5.6) - 80 = 60$ ft/sec.

To find where the magnitude of the velocity is **greater** than 60 ft/sec, check some arbitrary values in each of the following time intervals: $t \leq 0.8$, $0.8 \leq t \leq 5.6$ and $t \geq 5.6$.

Check $t = 0.5$: $v = 25(0.5) - 80 = -67.5$ ft/sec

Check $t = 2$: $v = 25(2) - 80 = -30$ ft/sec

Check $t = 6$: $v = 25(6) - 80 = -70$ ft/sec

You can see that the magnitude of the velocity is greater than 60 ft/sec only when $t \geq 5.6$ or when $t \leq 0.8$.

The answer checks out.

Further Resources

For a multimedia presentation on absolute value equations and inequalities, see http://www.teachertube.com/viewVideo.php?video_id=124516.

Lesson Summary

- The absolute value of a number is its distance from zero on a number line.
- $|x| = x$ if x is not negative, and $|x| = -x$ if x is negative.
- An equation or inequality with an absolute value in it **splits into two equations**, one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is **positive**, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is **negative**, then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type $|x| < a$ can be rewritten as “ $-a < x < a$.”
- Inequalities of the type $|x| > b$ can be rewritten as “ $x < -b$ or $x > b$.”

Review Questions

Evaluate the absolute values.

1. $|250|$
2. $|-12|$
3. $\left|-\frac{2}{5}\right|$
4. $\left|\frac{1}{10}\right|$

Find the distance between the points.

5. 12 and -11
6. 5 and 22
7. -9 and -18
8. -2 and 3

Solve the absolute value equations and interpret the results by graphing the solutions on the number line.

9. $|x - 5| = 10$
10. $|x + 2| = 6$
11. $|5x - 2| = 3$
12. $|x - 4| = -3$

Graph the absolute value functions.

13. $y = |x + 3|$
14. $y = |x - 6|$
15. $y = |4x + 2|$
16. $y = \left|\frac{x}{3} - 4\right|$

Solve the following inequalities and show the solution graph.

13. $|x| \leq 6$
14. $|x| > 3.5$
15. $|x| < 12$
16. $|x| > 10$

17. $|7x| \geq 21$
18. $|x - 5| > 8$
19. $|x + 7| < 3$
20. $\left|x - \frac{3}{4}\right| \leq \frac{1}{2}$
21. $|2x - 5| \geq 13$
22. $|5x + 3| < 7$
23. $\left|\frac{x}{3} - 4\right| \leq 2$
24. $\left|\frac{2x}{7} + 9\right| > \frac{5}{7}$
25. (a) How many solutions does the inequality $|x| \leq 0$ have?
(b) How about the inequality $|x| \geq 0$?
26. A company manufactures rulers. Their 12-inch rulers pass quality control if they are within $\frac{1}{32}$ inches of the ideal length. What is the longest and shortest ruler that can leave the factory?
27. A three month old baby boy weighs an average of 13 pounds. He is considered healthy if he is at most 2.5 lbs. more or less than the average weight. Find the weight range that is considered healthy for three month old boys.

6.5 Linear Inequalities in Two Variables

Learning Objectives

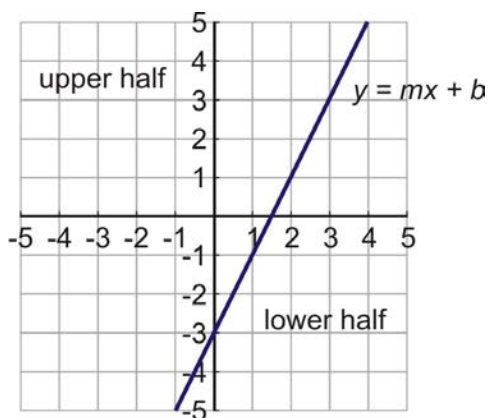
- Graph linear inequalities in one variable on the coordinate plane.
- Graph linear inequalities in two variables.
- Solve real-world problems using linear inequalities.

Introduction

Yasmeen is selling handmade bracelets for \$5 each and necklaces for \$7 each. How many of both does she need to sell to make at least \$100?

A **linear inequality** in two variables takes the form $y > mx + b$ or $y < mx + b$. Linear inequalities are closely related to graphs of straight lines; recall that a straight line has the equation $y = mx + b$.

When we graph a line in the coordinate plane, we can see that it divides the plane in half:



The solution to a linear inequality includes all the points in one half of the plane. We can tell which half by looking at the inequality sign:

- > The solution set is the half plane above the line.

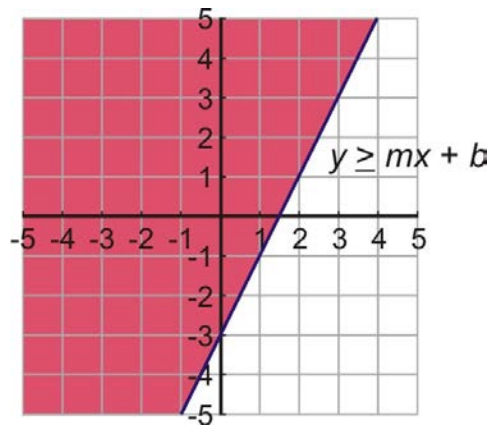
\geq The solution set is the half plane above the line and also all the points on the line.

$<$ The solution set is the half plane below the line.

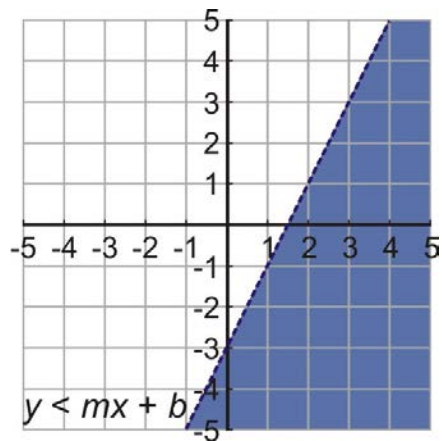
\leq The solution set is the half plane below the line and also all the points on the line.

For a strict inequality, we draw a **dashed line** to show that the points in the line *are not* part of the solution. For an inequality that includes the equals sign, we draw a **solid line** to show that the points on the line *are* part of the solution.

Here are some examples of linear inequality graphs. This is a graph of $y \geq mx + b$; the solution set is the line and the half plane above the line.



This is a graph of $y < mx + b$; the solution set is the half plane above the line, not including the line itself.



Graph Linear Inequalities in One Variable in the Coordinate Plane

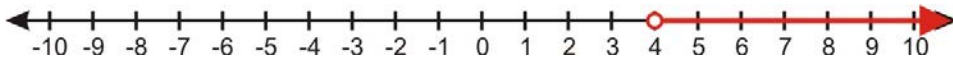
In the last few sections we graphed inequalities in one variable on the number line. We can also graph inequalities in one variable on the coordinate plane. We just need to remember that when we graph an equation of the type $x = a$ we get a vertical line, and when we graph an equation of the type $y = b$ we get a horizontal line.

Example 1

Graph the inequality $x > 4$ on the coordinate plane.

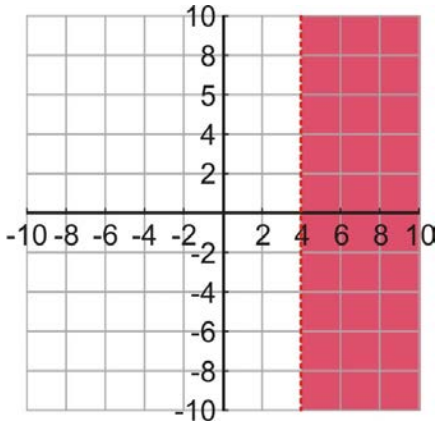
Solution

First let's remember what the solution to $x > 4$ looks like on the number line.



The solution to this inequality is the set of all real numbers x that are bigger than 4, not including 4. The solution is represented by a line.

In two dimensions, the solution still consists of all the points to the right of $x = 4$, but for all possible y -values as well. This solution is represented by the half plane to the right of $x = 4$. (You can think of it as being like the solution graphed on the number line, only stretched out vertically.)



The line $x = 4$ is dashed because the equals sign is not included in the inequality, meaning that points on the line are not included in the solution.

Example 2

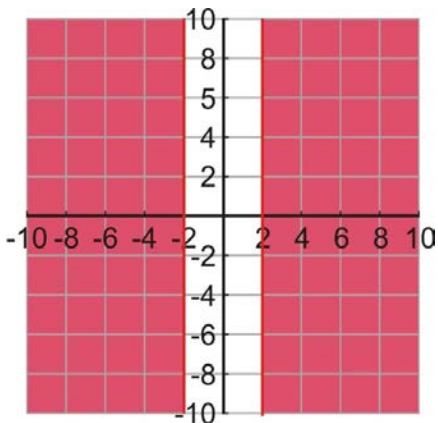
Graph the inequality $|x| \geq 2$.

Solution

The absolute value inequality $|x| \geq 2$ can be re-written as a compound inequality:

$$x \leq -2 \quad \text{or} \quad x \geq 2$$

In other words, the solution is all the coordinate points for which the value of x is smaller than or equal to -2 **or** greater than or equal to 2 . The solution is represented by the plane to the left of the vertical line $x = -2$ and the plane to the right of line $x = 2$.



Both vertical lines are solid because points on the lines are included in the solution.

Example 3

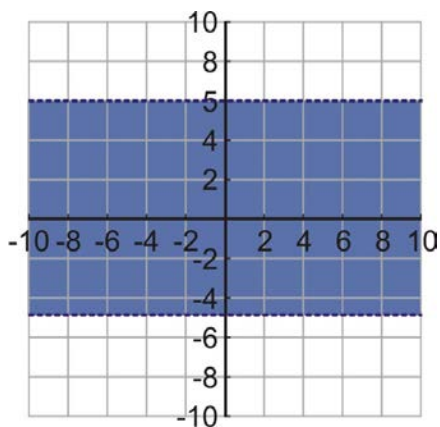
Graph the inequality $|y| < 5$

Solution

The absolute value inequality $|y| < 5$ can be re-written as $-5 < y < 5$. This is a compound inequality which can be expressed as

$$y > -5 \quad \text{and} \quad y < 5$$

In other words, the solution is all the coordinate points for which the value of y is larger than -5 **and** smaller than 5 . The solution is represented by the plane between the horizontal lines $y = -5$ and $y = 5$.



Both horizontal lines are dashed because points on the lines are not included in the solution.

Graph Linear Inequalities in Two Variables

The general procedure for graphing inequalities in two variables is as follows:

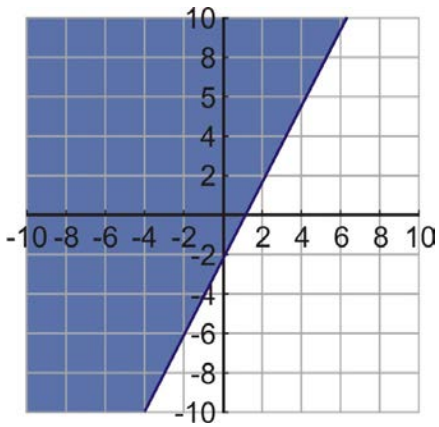
1. Re-write the inequality in slope-intercept form: $y = mx + b$. Writing the inequality in this form lets you know the direction of the inequality.
2. Graph the line of the equation $y = mx + b$ using your favorite method (plotting two points, using slope and y -intercept, using y -intercept and another point, or whatever is easiest). Draw the line as a dashed line if the equals sign is not included and a solid line if the equals sign is included.
3. Shade the half plane above the line if the inequality is “greater than.” Shade the half plane under the line if the inequality is “less than.”

Example 4

Graph the inequality $y \geq 2x - 3$.

Solution

The inequality is already written in slope-intercept form, so it's easy to graph. First we graph the line $y = 2x - 3$; then we shade the half-plane above the line. The line is solid because the inequality includes the equals sign.



Example 5

Graph the inequality $5x - 2y > 4$.

Solution

First we need to rewrite the inequality in slope-intercept form:

$$\begin{aligned} -2y &> -5x + 4 \\ y &< \frac{5}{2}x - 2 \end{aligned}$$

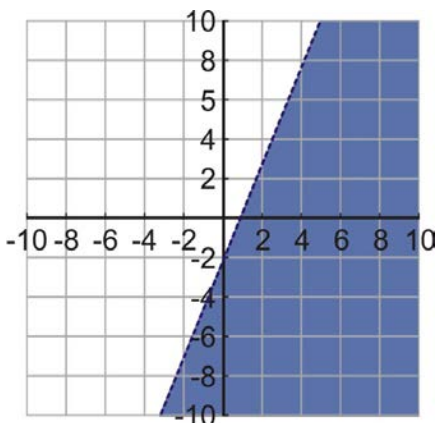
Notice that the inequality sign changed direction because we divided by a negative number.

To graph the equation, we can make a table of values:

Table 6.4:

x	y
-2	$\frac{5}{2}(-2) - 2 = -7$
0	$\frac{5}{2}(0) - 2 = -2$
2	$\frac{5}{2}(2) - 2 = 3$

After graphing the line, we shade the plane **below** the line because the inequality in slope-intercept form is **less than**. The line is dashed because the inequality does not include an equals sign.



Solve Real-World Problems Using Linear Inequalities

In this section, we see how linear inequalities can be used to solve real-world applications.

Example 8

A retailer sells two types of coffee beans. One type costs \$9 per pound and the other type costs \$7 per pound. Find all the possible amounts of the two different coffee beans that can be mixed together to get a quantity of coffee beans costing \$8.50 or less.

Solution

Let x = weight of \$9 per pound coffee beans in pounds.

Let y = weight of \$7 per pound coffee beans in pounds.

The cost of a pound of coffee blend is given by $9x + 7y$.

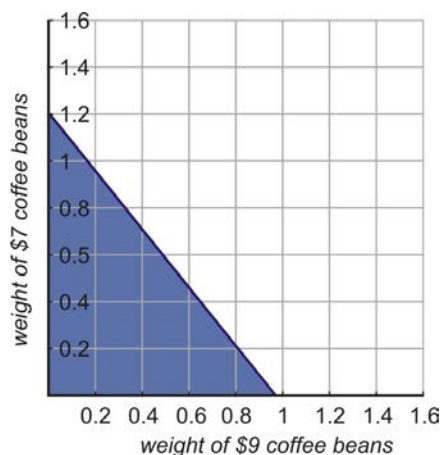
We are looking for the mixtures that cost \$8.50 or less. We write the inequality $9x + 7y \leq 8.50$.

Since this inequality is in standard form, it's easiest to graph it by finding the x - and y -intercepts. When $x = 0$, we have $7y = 8.50$ or $y = \frac{8.50}{7} \approx 1.21$. When $y = 0$, we have $9x = 8.50$ or $x = \frac{8.50}{9} \approx 0.94$. We can then graph the line that includes those two points.

Now we have to figure out which side of the line to shade. In y -intercept form, we shade the area **below** the line when the inequality is "less than." But in standard form that's not always true. We could convert the inequality to y -intercept form to find out which side to shade, but there is another way that can be easier.

The other method, which works for any linear inequality in any form, is to plug a random point into the inequality and see if it makes the inequality true. Any point that's not on the line will do; the point $(0, 0)$ is usually the most convenient.

In this case, plugging in 0 for x and y would give us $9(0) + 7(0) \leq 8.50$, which is true. That means we should shade the half of the plane that includes $(0, 0)$. If plugging in $(0, 0)$ gave us a false inequality, that would mean that the solution set is the part of the plane that does *not* contain $(0, 0)$.



Notice also that in this graph we show only the first quadrant of the coordinate plane. That's because weight values in the real world are always nonnegative, so points outside the first quadrant don't represent real-world solutions to this problem.

Example 9

Julius has a job as an appliance salesman. He earns a commission of \$60 for each washing machine he sells and \$130 for each refrigerator he sells. How many washing machines and refrigerators must Julius

sell in order to make \$1000 or more in commissions?

Solution

Let x = number of washing machines Julius sells.

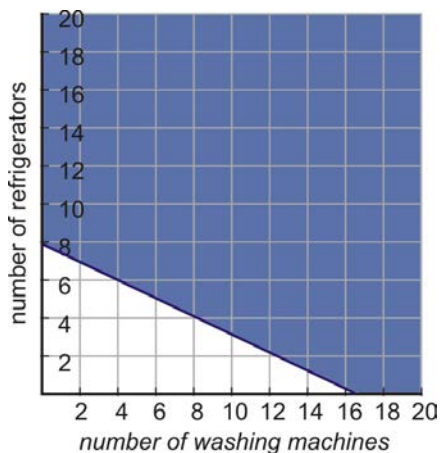
Let y = number of refrigerators Julius sells.

The total commission is $60x + 130y$.

We're looking for a total commission of \$1000 or more, so we write the inequality $60x + 130y \geq 1000$.

Once again, we can do this most easily by finding the x - and y -intercepts. When $x = 0$, we have $130y = 1000$, or $y = \frac{1000}{130} \approx 7.69$. When $y = 0$, we have $60x = 1000$, or $x = \frac{1000}{60} \approx 16.67$.

We draw a solid line connecting those points, and shade above the line because the inequality is "greater than." We can check this by plugging in the point $(0, 0)$: selling 0 washing machines and 0 refrigerators would give Julius a commission of \$0, which is *not* greater than or equal to \$1000, so the point $(0, 0)$ is *not* part of the solution; instead, we want to shade the side of the line that does *not* include it.



Notice also that we show only the first quadrant of the coordinate plane, because Julius's commission should be nonnegative.

The video at <http://www.youtube.com/watch?v=7629PsZLP1A&feature=related> contains more examples of real-world problems using inequalities in two variables.

Review Questions

Graph the following inequalities on the coordinate plane.

1. $x < 20$
2. $y \geq -5$
3. $|x| > 10$
4. $|y| \leq 7$
5. $y \leq 4x + 3$
6. $y > -\frac{x}{2} - 6$
7. $3x - 4y \geq 12$
8. $x + 7y < 5$
9. $6x + 5y > 1$
10. $y + 5 \leq -4x + 10$
11. $x - \frac{1}{2}y \geq 5$
12. $6x + y < 20$

13. $30x + 5y < 100$
14. Remember what you learned in the last chapter about families of lines.
 - (a) What do the graphs of $y > x + 2$ and $y < x + 5$ have in common?
 - (b) What do you think the graph of $x + 2 < y < x + 5$ would look like?
15. How would the answer to problem 6 change if you subtracted 2 from the right-hand side of the inequality?
16. How would the answer to problem 7 change if you added 12 to the right-hand side?
17. How would the answer to problem 8 change if you flipped the inequality sign?
18. A phone company charges 50 cents per minute during the daytime and 10 cents per minute at night. How many daytime minutes and nighttime minutes could you use in one week if you wanted to pay less than \$20?
19. Suppose you are graphing the inequality $y > 5x$.
 - (a) Why can't you plug in the point $(0, 0)$ to tell you which side of the line to shade?
 - (b) What happens if you do plug it in?
 - (c) Try plugging in the point $(0, 1)$ instead. Now which side of the line should you shade?
20. A theater wants to take in at least \$2000 for a certain matinee. Children's tickets cost \$5 each and adult tickets cost \$10 each.
 - (a) If x represents the number of adult tickets sold and y represents the number of children's tickets, write an inequality describing the number of tickets that will allow the theater to meet their minimum take.
 - (b) If 100 children's tickets and 100 adult tickets have already been sold, what inequality describes how many *more* tickets of both types the theater needs to sell?
 - (c) If the theater has only 300 seats (so only 100 are still available), what inequality describes the *maximum* number of additional tickets of both types the theater can sell?

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9616>.

Chapter 7

Solving Systems of Equations and Inequalities

7.1 Linear Systems by Graphing

Learning Objectives

- Determine whether an ordered pair is a solution to a system of equations.
- Solve a system of equations graphically.
- Solve a system of equations graphically with a graphing calculator.
- Solve word problems using systems of equations.

Introduction

In this lesson, we'll discover methods to determine if an ordered pair is a solution to a system of two equations. Then we'll learn to solve the two equations graphically and confirm that the solution is the point where the two lines intersect. Finally, we'll look at real-world problems that can be solved using the methods described in this chapter.

Determine Whether an Ordered Pair is a Solution to a System of Equations

A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

Consider this system of equations:

$$\begin{aligned}y &= x + 2 \\y &= -2x + 1\end{aligned}$$

Since the two lines are in a system, we deal with them together by graphing them on the same coordinate axes. We can use any method to graph them; let's do it by making a table of values for each line.

Line 1: $y = x + 2$

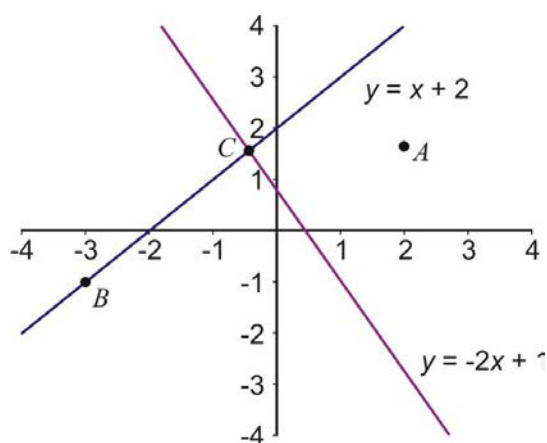
Table 7.1:

x	y
0	2
1	3

Line 2: $y = -2x + 1$

Table 7.2:

x	y
0	1
1	-1



We already know that any point that lies on a line is a solution to the equation for that line. That means that any point that lies on *both* lines in a system is a solution to both equations.

So in this system:

- Point A is not a solution to the system because it does not lie on either of the lines.
- Point B is not a solution to the system because it lies only on the blue line but not on the red line.
- Point C is a solution to the system because it lies on both lines at the same time.

In fact, point C is the only solution to the system, because it is the only point that lies on both lines. For a system of equations, the geometrical solution is the intersection of the two lines in the system. The algebraic solution is the ordered pair that solves both equations—in other words, the coordinates of that intersection point.

You can confirm the solution by plugging it into the system of equations, and checking that the solution works in each equation.

Example 1

Determine which of the points $(1, 3)$, $(0, 2)$, or $(2, 7)$ is a solution to the following system of equations:

$$y = 4x - 1$$

$$y = 2x + 3$$

Solution

To check if a coordinate point is a solution to the system of equations, we plug each of the x and y values into the equations to see if they work.

Point (1, 3):

$$\begin{aligned}y &= 4x - 1 \\3 & \stackrel{?}{=} \stackrel{?}{=} 4(1) - 1 \\3 & = 3 \text{ solution checks}\end{aligned}$$

$$\begin{aligned}y &= 2x + 3 \\3 & \stackrel{?}{=} \stackrel{?}{=} 2(1) + 3 \\3 & \neq 5 \text{ solution does not check}\end{aligned}$$

Point (1, 3) is on the line $y = 4x - 1$, but it is not on the line $y = 2x + 3$, so it is not a solution to the system.

Point (0, 2):

$$\begin{aligned}y &= 4x - 1 \\2 & \stackrel{?}{=} \stackrel{?}{=} 4(0) - 1 \\2 & \neq -1 \text{ solution does not check}\end{aligned}$$

Point (0, 2) is not on the line $y = 4x - 1$, so it is not a solution to the system. Note that it is not necessary to check the second equation because the point needs to be on both lines for it to be a solution to the system.

Point (2, 7):

$$\begin{aligned}y &= 4x - 1 \\7 & \stackrel{?}{=} \stackrel{?}{=} 4(2) - 1 \\7 & = 7 \text{ solution checks}\end{aligned}$$

$$\begin{aligned}y &= 2x + 3 \\7 & \stackrel{?}{=} \stackrel{?}{=} 2(2) + 3 \\7 & = 7 \text{ solution checks}\end{aligned}$$

Point (2, 7) is a solution to the system since it lies on both lines.

The solution to the system is the point (2, 7).

Determine the Solution to a Linear System by Graphing

The solution to a linear system of equations is the point, (if there is one) that lies on both lines. In other words, the solution is the point where the two lines intersect.

We can solve a system of equations by graphing the lines on the same coordinate plane and reading the intersection point from the graph.

This method most often offers only approximate solutions, so it's not sufficient when you need an exact answer. However, graphing the system of equations can be a good way to get a sense of what's really going on in the problem you're trying to solve, especially when it's a real-world problem.

Example 2

Solve the following system of equations by graphing:

$$y = 3x - 5$$

$$y = -2x + 5$$

Solution

Graph both lines on the same coordinate axis using any method you like.

In this case, let's make a table of values for each line.

Line 1: $y = 3x - 5$

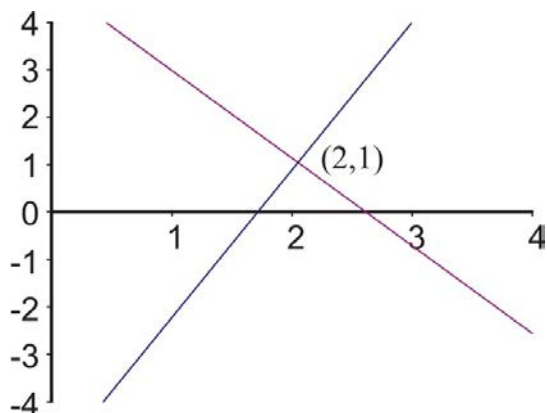
Table 7.3:

x	y
1	-2
2	1

Line 2: $y = -2x + 5$

Table 7.4:

x	y
1	3
2	1



The solution to the system is given by the intersection point of the two lines. The graph shows that the lines intersect at point $(2, 1)$. So **the solution is $x = 2, y = 1$ or $(2, 1)$.**

Example 3

Solve the following system of equations by graphing:

$$2x + 3y = 6$$

$$4x - y = -2$$

Solution

Since the equations are in standard form, this time we'll graph them by finding the x - and y -intercepts of each of the lines.

Line 1: $2x + 3y = 6$

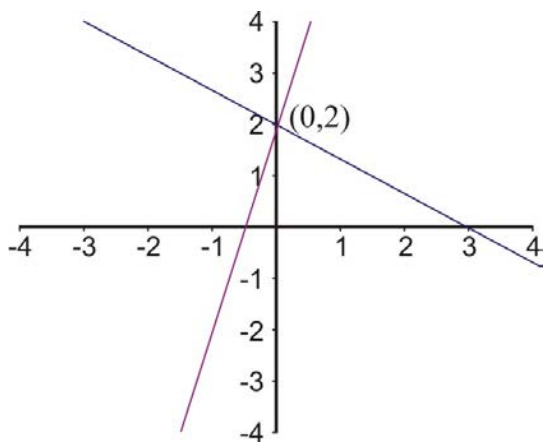
x -intercept: set $y = 0 \Rightarrow 2x = 6 \Rightarrow x = 3$ so the intercept is $(3, 0)$

y -intercept: set $x = 0 \Rightarrow 3y = 6 \Rightarrow y = 2$ so the intercept is $(0, 2)$

Line 2: $-4x + y = 2$

x -intercept: set $y = 0 \Rightarrow -4x = 2 \Rightarrow x = -\frac{1}{2}$ so the intercept is $(-\frac{1}{2}, 0)$

y -intercept: set $x = 0 \Rightarrow y = 2$ so the intercept is $(0, 2)$



The graph shows that the lines intersect at $(0, 2)$. Therefore, **the solution to the system of equations is $x = 0, y = 2$.**

Solving a System of Equations Using a Graphing Calculator

As an alternative to graphing by hand, you can use a graphing calculator to find or check solutions to a system of equations.

Example 4

Solve the following system of equations using a graphing calculator.

$$x - 3y = 4$$

$$2x + 5y = 8$$

To input the equations into the calculator, you need to rewrite them in slope-intercept form (that is, $y = mx + b$ form).

$$x - 3y = 4 \quad \Rightarrow \quad y = \frac{1}{3}x - \frac{4}{3}$$

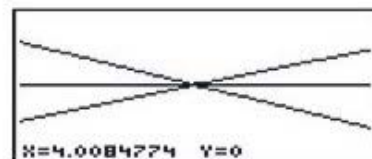
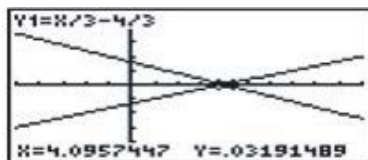
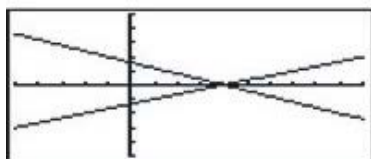
$$2x + 5y = 8 \quad \Rightarrow \quad y = -\frac{2}{5}x + \frac{8}{5}$$

Press the $[y=]$ button on the graphing calculator and enter the two functions as:

$$Y_1 = \frac{x}{3} - \frac{4}{3}$$

$$Y_2 = \frac{-2x}{5} + \frac{8}{5}$$

Now press **[GRAPH]**. Here's what the graph should look like on a TI-83 family graphing calculator with the window set to $-5 \leq x \leq 10$ and $-5 \leq y \leq 5$.



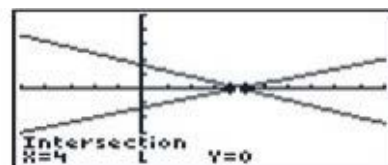
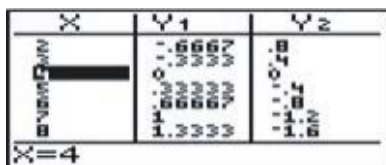
There are a few different ways to find the intersection point.

Option 1: Use **[TRACE]** and move the cursor with the arrows until it is on top of the intersection point. The values of the coordinate point will be shown on the bottom of the screen. The second screen above shows the values to be $X = 4.0957447$ and $Y = 0.03191489$.

Use the **[ZOOM]** function to zoom into the intersection point and find a more accurate result. The third screen above shows the system of equations after zooming in several times. A more accurate solution appears to be $X = 4$ and $Y = 0$.

Option 2 Look at the table of values by pressing **[2nd]** **[GRAPH]**. The first screen below shows a table of values for this system of equations. Scroll down until the Y -values for the two functions are the same. In this case this occurs at $X = 4$ and $Y = 0$.

(Use the **[TBLSET]** function to change the starting value for your table of values so that it is close to the intersection point and you don't have to scroll too long. You can also improve the accuracy of the solution by setting the value of Δ Table smaller.)



Option 3 Using the **[2nd]** **[TRACE]** function gives the second screen shown above.

Scroll down and select "intersect."

The calculator will display the graph with the question **[FIRSTCURVE]**? Move the cursor along the first curve until it is close to the intersection and press **[ENTER]**.

The calculator now shows **[SECONDCURVE]**?

Move the cursor to the second line (if necessary) and press **[ENTER]**.

The calculator displays **[GUESS]**?

Press **[ENTER]** and the calculator displays the solution at the bottom of the screen (see the third screen above).

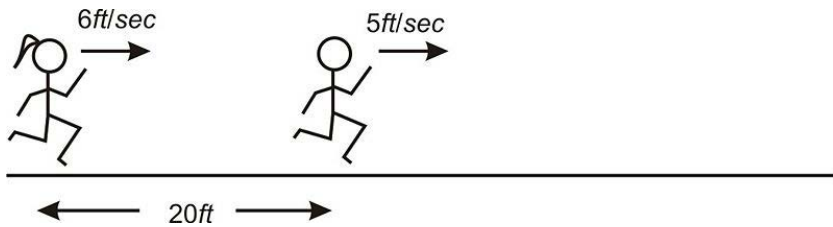
The point of intersection is $X = 4$ and $Y = 0$. Note that with this method, the calculator works out the intersection point for you, which is generally more accurate than your own visual estimate.

Solve Real-World Problems Using Graphs of Linear Systems

Consider the following problem:

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

Let's start by drawing a sketch. Here's what the race looks like when Nadia starts running; we'll call this time $t = 0$.



Now let's define two variables in this problem:

t = the time from when Nadia starts running

d = the distance of the runners from the starting point.

Since there are two runners, we need to write equations for each of them. That will be the *system of equations* for this problem.

For each equation, we use the formula: distance = speed \times time

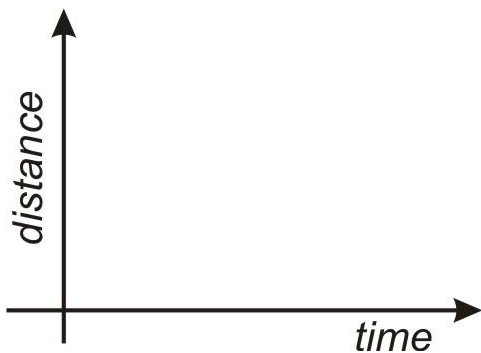
Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

(Remember that Peter was already 20 feet from the starting point when Nadia started running.)

Let's graph these two equations on the same coordinate axes.

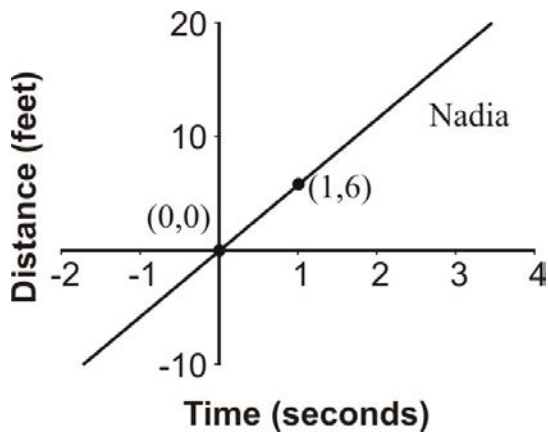
Time should be on the horizontal axis since it is the independent variable. Distance should be on the vertical axis since it is the dependent variable.



We can use any method for graphing the lines, but in this case we'll use the *slope-intercept* method since it makes more sense physically.

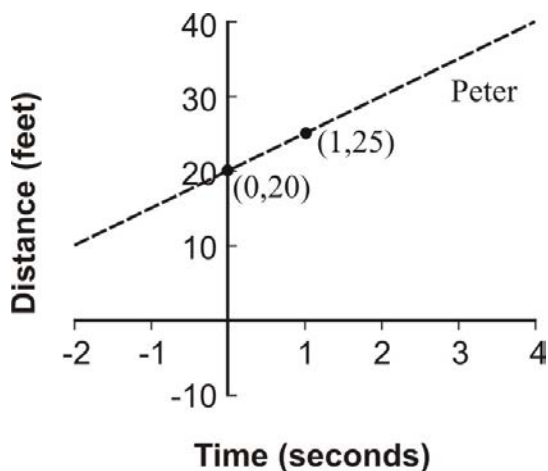
To graph the line that describes Nadia's run, start by graphing the y -intercept: $(0, 0)$. (If you don't see that this is the y -intercept, try plugging in the test-value of $x = 0$.)

The slope tells us that Nadia runs 6 feet every one second, so another point on the line is $(1, 6)$. Connecting these points gives us Nadia's line:

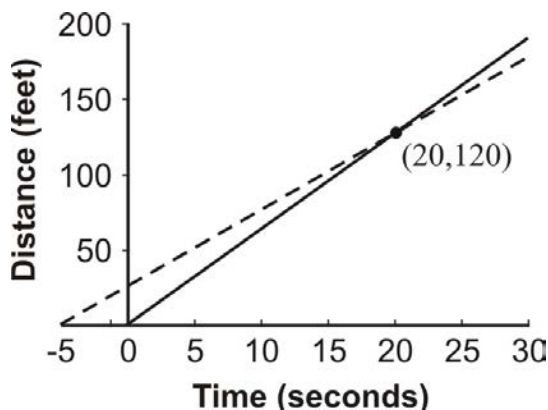


To graph the line that describes Peter's run, again start with the y-intercept. In this case this is the point (0, 20).

The slope tells us that Peter runs 5 feet every one second, so another point on the line is (1, 25). Connecting these points gives us Peter's line:



In order to find when and where Nadia and Peter meet, we'll graph both lines on the same graph and extend the lines until they cross. The crossing point is the solution to this problem.



The graph shows that Nadia and Peter meet **20 seconds after Nadia starts running, and 120 feet from the starting point.**

These examples are great at demonstrating that the solution to a system of linear equations means the

point at which the lines intersect. This is, in fact, the greatest strength of the graphing method because it offers a very visual representation of system of equations and its solution. You can also see, though, that finding the solution from a graph requires very careful graphing of the lines, and is really only practical when you're sure that the solution gives integer values for x and y . Usually, this method can only offer approximate solutions to systems of equations, so we need to use other methods to get an exact solution.

Review Questions

Determine which ordered pair satisfies the system of linear equations.

1. $y = 3x - 2$

$$y = -x$$

(a) $(1, 4)$

(b) $(2, 9)$

(c) $(\frac{1}{2}, \frac{-1}{2})$

2. $y = 2x - 3$

$$y = x + 5$$

(a) $(8, 13)$

(b) $(-7, 6)$

(c) $(0, 4)$

3. $2x + y = 8$

$$5x + 2y = 10$$

(a) $(-9, 1)$

(b) $(-6, 20)$

(c) $(14, 2)$

4. $3x + 2y = 6$

$$y = \frac{1}{2}x - 3$$

(a) $(3, \frac{-3}{2})$

(b) $(-4, 3)$

(c) $(\frac{1}{2}, 4)$

5. $2x - y = 10$

$$3x + y = -5$$

(a) $(4, -2)$

(b) $(1, -8)$

(c) $(-2, 5)$

Solve the following systems using the graphing method.

6. $y = x + 3$

$$y = -x + 3$$

7. $y = 3x - 6$

$$y = -x + 6$$

8. $2x = 4$

$$y = -3$$

9. $y = -x + 5$

$$-x + y = 1$$

10. $x + 2y = 8$
 $5x + 2y = 0$
11. $3x + 2y = 12$
 $4x - y = 5$
12. $5x + 2y = -4$
 $x - y = 2$
13. $2x + 4 = 3y$
 $x - 2y + 4 = 0$
14. $y = \frac{1}{2}x - 3$
 $2x - 5y = 5$
15. $y = 4$
 $x = 8 - 3y$
16. Try to solve the following system using the graphing method:
 $y = \frac{3}{5}x + 5$
 $y = -2x - \frac{1}{2}$.

- (a) What does it look like the x -coordinate of the solution should be?
- (b) Does that coordinate really give the same y -value when you plug it into both equations?
- (c) Why is it difficult to find the real solution to this system?

17. Try to solve the following system using the graphing method:
 $y = 4x + 8$
 $y = 5x + 1$.
Use a grid with x -values and y -values ranging from -10 to 10.

- (a) Do these lines appear to intersect?
- (b) Based on their equations, are they parallel?
- (c) What would we have to do to find their intersection point?

18. Try to solve the following system using the graphing method:
 $y = \frac{1}{2}x + 4$
 $y = \frac{4}{9}x + \frac{9}{2}$.
Use the same grid as before.

- (a) Can you tell exactly where the lines cross?
- (b) What would we have to do to make it clearer?

Solve the following problems by using the graphing method.

19. Mary's car has broken down and it will cost her \$1200 to get it fixed—or, for \$4500, she can buy a new, more efficient car instead. Her present car uses about \$2000 worth of gas per year, while gas for the new car would cost about \$1500 per year. After how many years would the total cost of fixing the car equal the total cost of replacing it?
20. Juan is considering two cell phone plans. The first company charges \$120 for the phone and \$30 per month for the calling plan that Juan wants. The second company charges \$40 for the same phone but charges \$45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
21. A tortoise and hare decide to race 30 feet. The hare, being much faster, decides to give the tortoise a 20 foot head start. The tortoise runs at 0.5 feet/sec and the hare runs at 5.5 feet per second. How long until the hare catches the tortoise?

7.2 Solving Linear Systems by Substitution

Learning Objectives

- Solve systems of equations with two variables by substituting for either variable.
- Manipulate standard form equations to isolate a single variable.
- Solve real-world problems using systems of equations.
- Solve mixture problems using systems of equations.

Introduction

In this lesson, we'll learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let's look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadia's equation: $d = 6t$

Peter's equation: $d = 5t + 20$

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersected—the point where the values for d and t satisfied **both** relationships. When the values for d and t are equal, that means that Peter and Nadia are at the same place at the same time.

But there's a faster way than graphing to solve this system of equations. Since we want the value of d to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for t . That is, if $d = 6t$ and $d = 5t + 20$, and the two d 's are equal to each other, then by the transitive property we have $6t = 5t + 20$. We can solve this for t :

$$\begin{array}{ll} 6t = 5t + 20 & \text{subtract } 5t \text{ from both sides :} \\ t = 20 & \text{substitute this value for } t \text{ into Nadia's equation :} \\ d = 6 \cdot 20 = 120 & \end{array}$$

Even if the equations weren't so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peter's equation to isolate t :

$$\begin{array}{ll} d = 5t + 20 & \text{subtract } 20 \text{ from both sides :} \\ d - 20 = 5t & \text{divide by } 5 : \\ \frac{d - 20}{5} = t & \end{array}$$

We can now **substitute** this expression for t into Nadia's equation ($d = 6t$) to solve:

$$d = 6\left(\frac{d - 20}{5}\right)$$

multiply both sides by 5 :

$$5d = 6(d - 20)$$

distribute the 6 :

$$5d = 6d - 120$$

subtract 6d from both sides :

$$-d = -120$$

divide by -1 :

$$d = 120$$

substitute value for d into our expression for t :

$$t = \frac{120 - 20}{5} = \frac{100}{5} = 20$$

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method**. In this lesson you'll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

Example 1

Let's look at an example where the equations are written in **standard form**.

Solve the system

$$\begin{aligned} 2x + 3y &= 6 \\ -4x + y &= 2 \end{aligned}$$

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of y is 1. So the easiest way to start is to use this equation to solve for y .

Solve the second equation for y :

$$\begin{aligned} -4x + y &= 2 && \textit{add 4x to both sides :} \\ y &= 2 + 4x \end{aligned}$$

Substitute this expression into the first equation:

$$\begin{aligned} 2x + 3(2 + 4x) &= 6 && \textit{distribute the 3 :} \\ 2x + 6 + 12x &= 6 && \textit{collect like terms :} \\ 14x + 6 &= 6 && \textit{subtract 6 from both sides :} \\ 14x &= 0 && \textit{and hence :} \\ x &= 0 \end{aligned}$$

Substitute back into our expression for y :

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution ($x = 0, y = 2$) that we found when we graphed these functions back in Lesson 7.1. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, let's look at a more complicated example. Here, the values of x and y we end up with aren't whole numbers, so they would be difficult to read off a graph!

Example 2

Solve the system

$$2x + 3y = 3$$

$$2x - 3y = -1$$

Again, we start by looking to isolate one variable in either equation. In this case it doesn't matter which equation we use—all the variables look about equally easy to solve for.

So let's solve the first equation for x :

$$2x + 3y = 3$$

$$2x = 3 - 3y$$

$$x = \frac{1}{2}(3 - 3y)$$

subtract 3y from both sides :

divide both sides by 2 :

Substitute this expression into the second equation:

$$2 \cdot \frac{1}{2}(3 - 3y) - 3y = -1$$

$$3 - 3y - 3y = -1$$

$$3 - 6y = -1$$

$$-6y = -4$$

$$y = \frac{2}{3}$$

cancel the fraction and re - write terms :

collect like terms :

subtract 3 from both sides :

divide by -6 :

Substitute into the expression we got for x :

$$x = \frac{1}{2} \left(3 - 3 \left(\frac{2}{3} \right) \right)$$
$$x = \frac{1}{2}$$

So our solution is $x = \frac{1}{2}, y = \frac{2}{3}$. You can see how the graphical solution $\left(\frac{1}{2}, \frac{2}{3}\right)$ might have been difficult to read accurately off a graph!

Solving Real-World Problems Using Linear Systems

Simultaneous equations can help us solve many real-world problems. We may be considering a purchase—for example, trying to decide whether it's cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but aren't sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Let's look at an example of that now.

Example 3

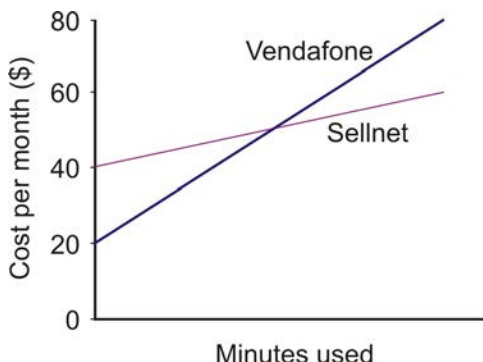
Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs \$20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges \$40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Anne's choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the *number of minutes* is the independent variable, it will be our x . Cost is *dependent* on minutes – the *cost per month* is the dependent variable and will be assigned y .

For Vendafone: $y = 0.25x + 20$

For Sellnet: $y = 0.08x + 40$

By writing the equations in slope-intercept form ($y = mx + b$), you can sketch a graph to visualize the situation:



The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line's slope). In order to help Anne decide which to choose, we'll find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for $y(0.25x + 20)$, we can substitute this expression directly into equation 2:

$$0.25x + 20 = 0.08x + 40$$

$$0.25x = 0.08x + 20$$

$$0.17x = 20$$

$$x = 117.65 \text{ minutes}$$

subtract 20 from both sides :

subtract 0.08x from both sides :

divide both sides by 0.17 :

rounded to 2 decimal places.

So if Anne uses 117.65 minutes a month (although she can't really do *exactly* that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, if Anne will use 117 minutes or less every month she should choose *Vendafone*. If she plans on using 118 or more minutes she should choose *Sellnet*.

Mixture Problems

Systems of equations crop up frequently in problems that deal with mixtures of two things—chemicals in a solution, nuts and raisins, or even the change in your pocket! Let's look at some examples of these.

Example 4

Janine empties her purse and finds that it contains only nickels (worth 5 cents each) and dimes (worth 10 cents each). If she has a total of 7 coins and they have a combined value of 45 cents, how many of each coin does she have?

Since we have 2 types of coins, let's call the number of nickels x and the number of dimes y . We are given two key pieces of information to make our equations: the number of coins and their value.

of coins equation: $x + y = 7$ (number of nickels) + (number of dimes)

value equation: $5x + 10y = 55$ (since nickels are worth 5c and dimes 10c)

We can quickly rearrange the first equation to isolate x :

$x = 7 - y$	<i>now substitute into equation 2 :</i>
$5(7 - y) + 10y = 55$	<i>distribute the 5 :</i>
$35 - 5y + 10y = 55$	<i>collect like terms :</i>
$35 + 5y = 55$	<i>subtract 35 from both sides :</i>
$5y = 20$	<i>divide by 5 :</i>
$\underline{y = 4}$	<i>substitute back into equation 1 :</i>
$x + 4 = 7$	<i>subtract 4 from both sides :</i>
$\underline{x = 3}$	

Janine has 3 nickels and 4 dimes.

Sometimes a question asks you to determine (from concentrations) how much of a particular substance to use. The substance in question could be something like coins as above, or it could be a chemical in solution, or even heat. In such a case, you need to know the amount of whatever substance is in each part. There are several common situations where to get one equation you simply add two given quantities, but to get the second equation you need to use a **product**. Three examples are below.

Table 7.5:

Type of mixture	First equation	Second equation
Coins (items with \$ value)	total number of items ($n_1 + n_2$)	total value (item value \times no. of items)
Chemical solutions	total solution volume ($V_1 + V_2$)	amount of solute (vol \times concentration)
Density of two substances	total amount or volume of mix	total mass (volume \times density)

For example, when considering mixing chemical solutions, we will most likely need to consider the total amount of *solute* in the individual parts and in the final mixture. (A solute is the chemical that is dissolved in a solution. An example of a solute is salt when added to water to make a brine.) To find the total amount, simply multiply the amount of the mixture by the **fractional concentration**. To illustrate, let's look at an example where you are given amounts relative to the whole.

Example 5

A chemist needs to prepare 500 ml of copper-sulfate solution with a 15% concentration. She wishes to use a high concentration solution (60%) and dilute it with a low concentration solution (5%) in order to do this. How much of each solution should she use?

Solution

To set this problem up, we first need to define our variables. Our unknowns are the amount of concentrated solution (x) and the amount of dilute solution (y). We will also convert the percentages (60%, 15% and 5%) into decimals (0.6, 0.15 and 0.05). The two pieces of critical information are the final volume (500 ml) and the final amount of solute (15% of 500 ml = 75 ml). Our equations will look like this:

Volume equation: $x + y = 500$

Solute equation: $0.6x + 0.05y = 75$

To isolate a variable for substitution, we can see it's easier to start with equation 1:

$$\begin{aligned}
x + y &= 500 \\
x &= 500 - y \\
0.6(500 - y) + 0.05y &= 75 \\
300 - 0.6y + 0.05y &= 75 \\
300 - 0.55y &= 75 \\
-0.55y &= -225 \\
\underline{y = 409 \text{ ml}} \\
x = 500 - 409 &= \underline{91 \text{ ml}}
\end{aligned}$$

subtract y from both sides :
now substitute into equation 2 :
distribute the 0.6 :
collect like terms :
subtract 300 from both sides :
divide both sides by -0.55 :
substitute back into equation for x :

So the chemist should mix 91 ml of the 60% solution with 409 ml of the 5% solution.

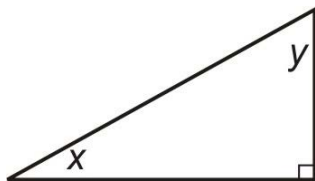
Further Practice

For lots more practice solving linear systems, check out this web page: <http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl>

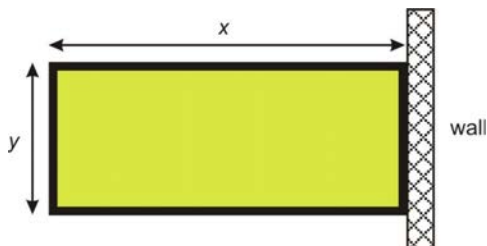
After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

Review Questions

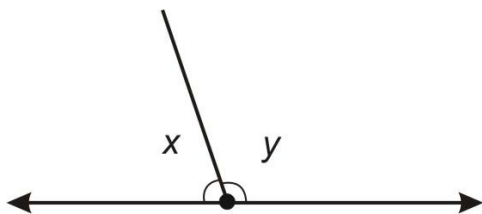
- Solve the system:
 $x + 2y = 9$
 $3x + 5y = 20$
- Solve the system:
 $x - 3y = 10$
 $2x + y = 13$
- Solve the system:
 $2x + 0.5y = -10$
 $x - y = -10$
- Solve the system:
 $2x + 0.5y = 3$
 $x + 2y = 8.5$
- Solve the system:
 $3x + 5y = -1$
 $x + 2y = -1$
- Solve the system:
 $3x + 5y = -3$
 $x + 2y = -\frac{4}{3}$
- Solve the system:
 $x - y = -\frac{12}{5}$
 $2x + 5y = -2$
- Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?



9. The sum of two numbers is 70. They differ by 11. What are the numbers?
10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?
11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?



12. A ray cuts a line forming two angles. The difference between the two angles is 18° . What does each angle measure?



13. I have \$15 and wish to buy five pounds of mixed nuts for a party. Peanuts cost \$2.20 per pound. Cashews cost \$4.70 per pound.
 - (a) How many pounds of each should I buy?
 - (b) If I suddenly realize I need to set aside \$5 to buy chips, can I still buy 5 pounds of nuts with the remaining \$10?
 - (c) What's the greatest amount of nuts I can buy?
14. A chemistry experiment calls for one liter of sulfuric acid at a 15% concentration, but the supply room only stocks sulfuric acid in concentrations of 10% and 35%.
 - (a) How many liters of each should be mixed to give the acid needed for the experiment?
 - (b) How many liters should be mixed to give *two* liters at a 15% concentration?
15. Bachellet wants to know the density of her bracelet, which is a mix of gold and silver. Density is total mass divided by total volume. The density of gold is 19.3 g/cc and the density of silver is 10.5 g/cc. The jeweler told her that the volume of silver in the bracelet was 10 cc and the volume of gold was 20 cc. Find the combined density of her bracelet.
16. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?
17. Tickets to a show cost \$10 in advance and \$15 at the door. If 120 tickets are sold for a total of \$1390, how many of the tickets were bought in advance?
18. The multiple-choice questions on a test are worth 2 points each, and the short-answer questions are worth 5 points each.
 - (a) If the whole test is worth 100 points and has 35 questions, how many of the questions are multiple-choice and how many are short-answer?

- (b) If Kwan gets 31 questions right and ends up with a score of 86 on the test, how many questions of each type did she get right? (Assume there is no partial credit.)
- (c) If Ashok gets 5 questions wrong and ends up with a score of 87 on the test, how many questions of each type did he get wrong? (Careful!)
- (d) What are two ways you could have set up the equations for part c?
- (e) How could you have set up part b differently?

7.3 Solving Linear Systems by Elimination

Learning Objectives

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve a linear system of equations by multiplication and then addition or subtraction.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Introduction

In this lesson, we'll see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns (x and y) to a single unknown (either x or y), this method is often referred to by *solving by elimination*. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let's look at the simple example of buying apples and bananas.

Example 1

If one apple plus one banana costs \$1.25 and one apple plus 2 bananas costs \$2.00, how much does one banana cost? One apple?

It shouldn't take too long to discover that each banana costs \$0.75. After all, the second purchase just contains 1 more banana than the first, and costs \$0.75 more, so that one banana must cost \$0.75.

Here's what we get when we describe this situation with algebra:

$$\begin{aligned}a + b &= 1.25 \\a + 2b &= 2.00\end{aligned}$$

Now we can subtract the number of apples and bananas in the first equation from the number in the second equation, and also subtract the cost in the first equation from the cost in the second equation, to get the *difference* in cost that corresponds to the *difference* in items purchased.

$$(a + 2b) - (a + b) = 2.00 - 1.25 \rightarrow b = 0.75$$

That gives us the cost of one banana. To find out how much one apple costs, we subtract \$0.75 from the total cost of one apple and one banana.

$$a + 0.75 = 1.25 \rightarrow a = 1.25 - 0.75 \rightarrow a = 0.50$$

So an apple costs 50 cents.

To solve systems using addition and subtraction, we'll be using exactly this idea – by looking at the *sum* or *difference* of the two equations we can determine a value for one of the unknowns.

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example 2

Solve this system by addition:

$$\begin{aligned}3x + 2y &= 11 \\5x - 2y &= 13\end{aligned}$$

Solution

We will add **everything** on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right:

$$(3x + 2y) + (5x - 2y) = 11 + 13 \rightarrow 8x = 24 \rightarrow x = 3$$

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you **MUST** be sure to keep the x 's and y 's in their own columns. You may also wish to use terms like "0y" as a placeholder!

$$\begin{array}{r}3x + 2y = 11 \\+ (5x - 2y) = 13 \\ \hline8x + 0y = 24\end{array}$$

Again we get $8x = 24$, or $x = 3$. To find a value for y , we simply substitute our value for x back in.

Substitute $x = 3$ into the second equation:

$$\begin{aligned}5 \cdot 3 - 2y &= 13 && \text{since } 5 \times 3 = 15, \text{ we subtract 15 from both sides :} \\-2y &= -2 && \text{divide by } -2 \text{ to get :} \\y &= 1\end{aligned}$$

The reason this method worked is that the y -coefficients of the two equations were opposites of each other: 2 and -2. Because they were opposites, they canceled each other out when we added the two equations together, so our final equation had no y -term in it and we could just solve it for x .

In a little while we'll see how to use the addition method when the coefficients are not opposites, but for now let's look at another example where they are.

Example 3

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, how fast is the current? How fast would Andrew travel in calm water?

Solution

First we convert our problem into equations. We have two unknowns to solve for, so we'll call the speed that Andrew paddles at x , and the speed of the river y . When traveling downstream, Andrew's speed is boosted by the river current, so his total speed is his paddling speed *plus* the speed of the river ($x + y$). Traveling upstream, the river is working against him, so his total speed is his paddling speed *minus* the speed of the river ($x - y$).

Downstream Equation: $x + y = 7$

Upstream Equation: $x - y = 1.5$

Next we'll eliminate one of the variables. If you look at the two equations, you can see that the coefficient of y is $+1$ in the first equation and -1 in the second. Clearly $(+1) + (-1) = 0$, so this is the variable we will eliminate. To do this we simply add equation 1 to equation 2. We must be careful to collect like terms, and make sure that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x + y) + (x - y) = 7 + 1.5 \Rightarrow 2x = 8.5 \Rightarrow x = 4.25$$

Or, using the column method we used in example 2:

$$\begin{array}{r} x + y = 7 \\ + x - y = 1.5 \\ \hline 2x + 0y = 8.5 \end{array}$$

Again we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for y , we plug our value for x into either equation and isolate our unknown. In this example, we'll plug it into the first equation:

$$\begin{array}{r} 4.25 + y = 7 \\ y = 2.75 \end{array} \qquad \text{subtract 4.25 from both sides :}$$

Andrew paddles at 4.25 miles per hour. The river moves at 2.75 miles per hour.

Solving Linear Systems Using Subtraction of Equations

Another, very similar method for solving systems is subtraction. When the x - or y -coefficients in both equations are the same (including the sign) instead of being opposites, you can **subtract** one equation from the other.

If you look again at Example 3, you can see that the coefficient for x in both equations is $+1$. Instead of adding the two equations together to get rid of the y 's, you could have subtracted to get rid of the x 's:

$$\begin{array}{r} (x + y) - (x - y) = 7 - 1.5 \Rightarrow 2y = 5.5 \Rightarrow y = 2.75 \\ \text{or...} \\ x + y = 7 \\ - (x - y) = -1.5 \\ \hline 0x + 2y = 5.5 \end{array}$$

So again we get $y = 2.75$, and we can plug that back in to determine x .

The method of subtraction is just as straightforward as addition, so long as you remember the following:

- Always put the equation you are subtracting in parentheses, and distribute the negative.
- Don't forget to **subtract** the numbers on the right-hand side.
- Always remember that subtracting a negative is the same as adding a positive.

Example 4

Peter examines the coins in the fountain at the mall. He counts 107 coins, all of which are either pennies or nickels. The total value of the coins is \$3.47. How many of each coin did he see?

Solution

We have 2 types of coins, so let's call the number of pennies x and the number of nickels y . The total value of all the pennies is just x , since they are worth 1¢ each. The total value of the nickels is $5y$. We are given two key pieces of information to make our equations: the number of coins and their value in cents.

$$\begin{array}{lll} \# \text{ of coins equation :} & x + y = 107 & (\text{number of pennies}) + (\text{number of nickels}) \\ \text{value equation :} & x + 5y = 347 & \text{pennies are worth 1¢, nickels are worth 5¢.} \end{array}$$

We'll jump straight to subtracting the two equations:

$$\begin{array}{r} x + y = 107 \\ - (x + 5y) = -347 \\ \hline -4y = -240 \\ y = 60 \end{array}$$

Substituting this value back into the first equation:

$$\begin{array}{ll} x + 60 = 107 & \text{subtract 60 from both sides :} \\ x = 47 & \end{array}$$

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth \$3.00) making a total of \$3.47.

Solving Linear Systems Using Multiplication

So far, we've seen that the elimination method works well when the coefficient of one variable happens to be the same (or opposite) in the two equations. But what if the two equations don't have any coefficients the same?

It turns out that we can still use the elimination method; we just have to *make* one of the coefficients match. We can accomplish this by multiplying one or both of the equations by a constant.

Here's a quick review of how to do that. Consider the following questions:

1. If 10 apples cost \$5, how much would 30 apples cost?
2. If 3 bananas plus 2 carrots cost \$4, how much would 6 bananas plus 4 carrots cost?

If you look at the first equation, it should be obvious that each apple costs \$0.50. So 30 apples should cost \$15.00.

The second equation is trickier; it isn't obvious what the individual price for either bananas or carrots is. Yet we know that the answer to question 2 is \$8.00. How?

If we look again at question 1, we see that we can write an equation: $10a = 5$ (a being the cost of 1 apple). So to find the cost of 30 apples, we *could* solve for a and then multiply by 30—but we could also just multiply both sides of the equation by 3. We would get $30a = 15$, and that tells us that 30 apples cost \$15.

And we can do the same thing with the second question. The equation for this situation is $3b + 2c = 4$, and we can see that we need to solve for $(6b + 4c)$, which is simply 2 times $(3b + 2c)$! So algebraically, we are simply multiplying the entire equation by 2:

$$\begin{array}{l} 2(3b + 2c) = 2 \cdot 4 \\ 6b + 4c = 8 \end{array} \qquad \text{distribute and multiply :}$$

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

Solving a Linear System by Multiplying One Equation

If we can multiply every term in an equation by a fixed number (a **scalar**), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

Example 5

Solve the system:

$$\begin{array}{l} 7x + 4y = 17 \\ 5x - 2y = 11 \end{array}$$

Solution

You can easily see that if we multiply the second equation by 2, the coefficients of y will be $+4$ and -4 , allowing us to solve the system by addition:

2 times equation 2:

$$\begin{array}{l} 10x - 4y = 22 \\ + (7x + 4y) = 17 \\ \hline 17x = 34 \end{array} \qquad \text{now add to equation one :}$$

$$\text{divide by 17 to get :} \qquad x = 2$$

Now simply substitute this value for x back into equation 1:

$$\begin{array}{l} 7 \cdot 2 + 4y = 17 \\ 4y = 3 \\ y = 0.75 \end{array} \qquad \begin{array}{l} \text{since } 7 \times 2 = 14, \text{ subtract 14 from both sides :} \\ \text{divide by 4 :} \end{array}$$

Example 6

Anne is rowing her boat along a river. Rowing downstream, it takes her 2 minutes to cover 400 yards. Rowing upstream, it takes her 8 minutes to travel the same 400 yards. If she was rowing equally hard in

both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

Solution

Step one: first we convert our problem into equations. We know that *distance traveled* is equal to *speed* \times *time*. We have two unknowns, so we'll call the speed of the river x , and the speed that Anne rows at y . When traveling downstream, her total speed is her rowing speed plus the speed of the river, or $(x + y)$. Going upstream, her speed is hindered by the speed of the river, so her speed upstream is $(x - y)$.

Downstream Equation: $2(x + y) = 400$

Upstream Equation: $8(x - y) = 400$

Distributing gives us the following system:

$$2x + 2y = 400$$

$$8x - 8y = 400$$

Right now, we can't use the method of elimination because none of the coefficients match. But if we multiplied the top equation by 4, the coefficients of y would be $+8$ and -8 . Let's do that:

$$\begin{array}{r} 8x + 8y = 1,600 \\ + (8x - 8y) = 400 \\ \hline 16x = 2,000 \end{array}$$

Now we divide by 16 to obtain $x = 125$.

Substitute this value back into the first equation:

$$2(125 + y) = 400$$

$$125 + y = 200$$

$$y = 75$$

divide both sides by 2 :

subtract 125 from both sides :

Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

Solving a Linear System by Multiplying Both Equations

So what do we do if none of the coefficients match and none of them are simple multiples of each other? We do the same thing we do when we're adding fractions whose denominators aren't simple multiples of each other. Remember that when we add fractions, we have to find a **lowest common denominator**—that is, the lowest common multiple of the two denominators—and sometimes we have to rewrite not just one, but both fractions to get them to have a common denominator. Similarly, sometimes we have to multiply both equations by different constants in order to get one of the coefficients to match.

Example 7

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2060 miles in five days. Anne travels 880 miles from Norfolk, Virginia, and it takes her three days. If Anne pays \$840 and Andrew pays \$1845, what does I-Haul charge

a) *per day?*

b) *per mile traveled?*

Solution

First, we'll set up our equations. Again we have 2 unknowns: the **daily rate** (we'll call this x), and the **per-mile rate** (we'll call this y).

Anne's equation: $3x + 880y = 840$

Andrew's Equation: $5x + 2060y = 1845$

We can't just multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of x (as they are easier to deal with than the coefficients of y), we see that they both have a common multiple of 15 (in fact 15 is the ***lowest common multiple***). So we can multiply both equations.

Multiply the top equation by 5:

$$15x + 4400y = 4200$$

Multiply the lower equation by 3:

$$15x + 6180y = 5535$$

Subtract:

$$\begin{array}{r} 15x + 4400y = 4200 \\ - (15x + 6180y) = 5535 \\ \hline -1780y = -1335 \end{array}$$

$$\text{Divide by } -1780 : y = 0.75$$

Substitute this back into the top equation:

$$\begin{array}{ll} 3x + 880(0.75) = 840 & \text{since } 880 \times 0.75 = 660, \text{ subtract } 660 \text{ from both sides :} \\ 3x = 180 & \text{divide both sides by } 3 \\ x = 60 & \end{array}$$

I-Haul charges \$60 per day plus \$0.75 per mile.

Comparing Methods for Solving Linear Systems

Now that we've covered the major methods for solving linear equations, let's review them. For simplicity, we'll look at them in table form. This should help you decide which method would be best for a given situation.

Table 7.6:

Method:	Best used when you...	Advantages:	Comment:
Graphing	...don't need an accurate answer.	Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don't have to do any computation.	Can lead to imprecise answers with non-integer solutions.
Substitution	...have an <i>explicit</i> equation for one variable (e.g. $y = 14x + 2$)	Works on all systems. Reduces the system to one variable, making it easier to solve.	You are not often given explicit functions in systems problems, so you may have to do extra work to get one of the equations into that form.
Elimination by Addition or Subtraction	...have matching coefficients for one variable in both equations.	Easy to combine equations to eliminate one variable. Quick to solve.	It is not very likely that a given system will have matching coefficients.
Elimination by Multiplication and then Addition and Subtraction	...do not have any variables defined explicitly or any matching coefficients.	Works on all systems. Makes it possible to combine equations to eliminate one variable.	Often more algebraic manipulation is needed to prepare the equations.

The table above is only a guide. You might prefer to use the graphical method for every system in order to better understand what is happening, or you might prefer to use the multiplication method even when a substitution would work just as well.

Example 8

Two angles are **complementary** when the sum of their angles is 90° . Angles A and B are complementary angles, and twice the measure of angle A is 9° more than three times the measure of angle B . Find the measure of each angle.

Solution

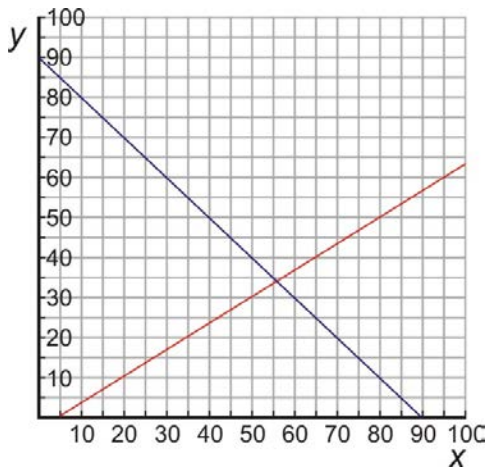
First we write out our 2 equations. We will use x to be the measure of angle A and y to be the measure of angle B . We get the following system:

$$\begin{aligned}x + y &= 90 \\2x &= 3y + 9\end{aligned}$$

First, we'll solve this system with the graphical method. For this, we need to convert the two equations to $y = mx + b$ form:

$$\begin{aligned}x + y = 90 &\Rightarrow y = -x + 90 \\2x = 3y + 9 &\Rightarrow y = \frac{2}{3}x - 3\end{aligned}$$

The first line has a slope of -1 and a y-intercept of 90, and the second line has a slope of $\frac{2}{3}$ and a y-intercept of -3. The graph looks like this:



In the graph, it appears that the lines cross at around $x = 55, y = 35$, but it is difficult to tell exactly! Graphing by hand is not the best method in this case!

Next, we'll try solving by substitution. Let's look again at the system:

$$\begin{aligned}x + y &= 90 \\2x &= 3y + 9\end{aligned}$$

We've already seen that we can start by solving either equation for y , so let's start with the first one:

$$y = 90 - x$$

Substitute into the second equation:

$$\begin{aligned}2x &= 3(90 - x) + 9 && \text{distribute the 3 :} \\2x &= 270 - 3x + 9 && \text{add 3x to both sides :} \\5x &= 270 + 9 = 279 && \text{divide by 5 :} \\x &= 55.8^\circ\end{aligned}$$

Substitute back into our expression for y :

$$y = 90 - 55.8 = 34.2^\circ$$

Angle A measures 55.8° ; angle B measures 34.2° .

Finally, we'll try solving by elimination (with multiplication):

Rearrange equation one to standard form:

$$x + y = 90 \quad \Rightarrow \quad 2x + 2y = 180$$

Multiply equation two by 2:

$$2x = 3y + 9 \quad \Rightarrow \quad 2x - 3y = 9$$

Subtract:

$$\begin{array}{r}
 2x + 2y = 180 \\
 - (2x - 3y) = -9 \\
 \hline
 5y = 171
 \end{array}$$

Divide by 5 to obtain $y = 34.2^\circ$

Substitute this value into the very first equation:

$$\begin{array}{ll}
 x + 34.2 = 90 & \text{subtract } 34.2 \text{ from both sides :} \\
 x = 55.8^\circ &
 \end{array}$$

Angle A measures 55.8° ; angle B measures 34.2° .

Even though this system looked ideal for substitution, the method of multiplication worked well too. Once the equations were rearranged properly, the solution was quick to find. You'll need to decide yourself which method to use in each case you see from now on. Try to master all the techniques, and recognize which one will be most efficient for each system you are asked to solve.

The following Khan Academy video contains three examples of solving systems of equations using addition and subtraction as well as multiplication (which is the next topic): <http://www.youtube.com/watch?v=nok99J0hcjo> (9:57). (Note that the narrator is not always careful about showing his work, and you should try to be neater in your mathematical writing.)

For even more practice, we have this video. One common type of problem involving systems of equations (especially on standardized tests) is "age problems." In the following video the narrator shows two examples of age problems, one involving a single person and one involving two people. [Khan Academy Age Problems \(7:13\)](#)

Review Questions

- Solve the system:
 $3x + 4y = 2.5$
 $5x - 4y = 25.5$
- Solve the system:
 $5x + 7y = -31$
 $5x - 9y = 17$
- Solve the system:
 $3y - 4x = -33$
 $5x - 3y = 40.5$
- Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for \$2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs \$1.79. What is the cost of a candy bar and a fruit roll-up individually?
- A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.

6. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs \$14.29 and a 17-mile journey costs \$19.91, calculate:
 - (a) the pick-up fee
 - (b) the per-mile rate
 - (c) the cost of a seven mile trip
7. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a 7-minute call costs \$4.25 and a 12-minute call costs \$5.50, find each rate.
8. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns \$35 per hour, and the builder earns \$28 per hour. Together they were paid \$330.75, but the plumber earned \$106.75 more than the builder. How many hours did each work?
9. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned \$220. In his second week, he managed to sell 13 warranties and earned \$280. What is Paul's hourly rate, and how much extra does he get for selling each warranty?

Solve the following systems using multiplication.

10. $5x - 10y = 15$
 $3x - 2y = 3$
11. $5x - y = 10$
 $3x - 2y = -1$
12. $5x + 7y = 15$
 $7x - 3y = 5$
13. $9x + 5y = 9$
 $12x + 8y = 12.8$
14. $4x - 3y = 1$
 $3x - 4y = 4$
15. $7x - 3y = -3$
 $6x + 4y = 3$

Solve the following systems using any method.

16. $x = 3y$
 $x - 2y = -3$
17. $y = 3x + 2$
 $y = -2x + 7$
18. $5x - 5y = 5$
 $5x + 5y = 35$
19. $y = -3x - 3$
 $3x - 2y + 12 = 0$
20. $3x - 4y = 3$
 $4y + 5x = 10$
21. $9x - 2y = -4$
 $2x - 6y = 1$
22. Supplementary angles are two angles whose sum is 180° . Angles A and B are supplementary angles. The measure of Angle A is 18° less than twice the measure of Angle B . Find the measure of each angle.

23. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?
24. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less than twice the length of the second piece, what are the lengths of the two pieces?
25. Mr. Stein invested a total of \$100,000 in two companies for a year. Company A's stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?
26. A baker sells plain cakes for \$7 and decorated cakes for \$11. On a busy Saturday the baker started with 120 cakes, and sold all but three. His takings for the day were \$991. How many plain cakes did he sell that day, and how many were decorated before they were sold?
27. Twice John's age plus five times Claire's age is 204. Nine times John's age minus three times Claire's age is also 204. How old are John and Claire?

7.4 Special Types of Linear Systems

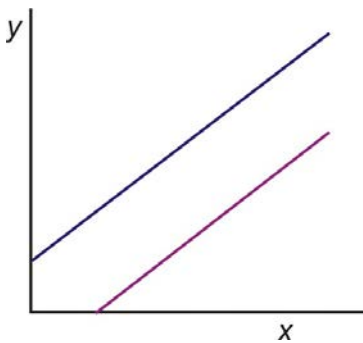
Learning Objectives

- Identify and understand what is meant by an **inconsistent linear system**.
- Identify and understand what is meant by a **consistent linear system**.
- Identify and understand what is meant by a **dependent linear system**.

Introduction

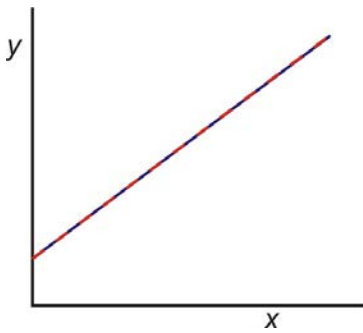
As we saw in Section 7.1, a system of linear equations is a set of linear equations which must be solved together. The lines in the system can be graphed together on the same coordinate graph and the solution to the system is the point at which the two lines intersect.

Or at least that's what usually happens. But what if the lines turn out to be parallel when we graph them?



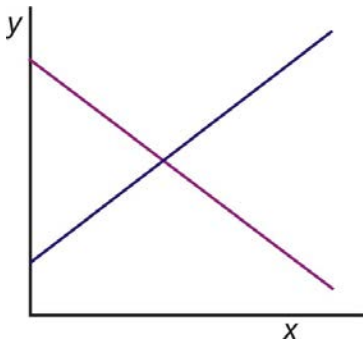
If the lines are parallel, they won't ever intersect. That means that the system of equations they represent has no solution. A system with no solutions is called an **inconsistent system**.

And what if the lines turn out to be identical?



If the two lines are the same, then *every* point on one line is also on the other line, so every point on the line is a solution to the system. The system has an **infinite number** of solutions, and the two equations are really just different forms of the same equation. Such a system is called a **dependent system**.

But usually, two lines cross at exactly one point and the system has exactly one solution:



A system with exactly one solution is called a **consistent system**.

To identify a system as **consistent**, **inconsistent**, or **dependent**, we can graph the two lines on the same graph and see if they intersect, are parallel, or are the same line. But sometimes it is hard to tell whether two lines are parallel just by looking at a roughly sketched graph.

Another option is to write each line in slope-intercept form and compare the slopes and y -intercepts of the two lines. To do this we must remember that:

- Lines with different slopes always intersect.
- Lines with the same slope but different y -intercepts are parallel.
- Lines with the same slope and the same y -intercepts are identical.

Example 1

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$2x - 5y = 2$$

$$4x + y = 5$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lcl}
 2x - 5y = 2 & \Rightarrow & -5y = -2x + 2 & \Rightarrow & y = \frac{2}{5}x - \frac{2}{5} \\
 4x + y = 5 & & y = -4x + 5 & & y = -4x + 5
 \end{array}$$

The slopes of the two equations are different; therefore the lines must cross at a single point and the system has exactly one solution. This is a **consistent system**.

Example 2

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}3x &= 5 - 4y \\6x + 8y &= 7\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lcl}3x = 5 - 4y & \Rightarrow & 4y = -3x + 5 & \Rightarrow & y = -\frac{3}{4}x + \frac{5}{4} \\6x + 8y = 7 & \Rightarrow & 8y = -6x + 7 & \Rightarrow & y = -\frac{3}{4}x + \frac{7}{8}\end{array}$$

The slopes of the two equations are the same but the y -intercepts are different; therefore the lines are parallel and the system has no solutions. This is an **inconsistent system**.

Example 3

Determine whether the following system has exactly one solution, no solutions, or an infinite number of solutions.

$$\begin{aligned}x + y &= 3 \\3x + 3y &= 9\end{aligned}$$

Solution

We must rewrite the equations so they are in slope-intercept form

$$\begin{array}{lcl}x + y = 3 & \Rightarrow & y = -x + 3 & \Rightarrow & y = -x + 3 \\3x + 3y = 9 & \Rightarrow & 3y = -3x + 9 & \Rightarrow & y = -x + 3\end{array}$$

The lines are identical; therefore the system has an infinite number of solutions. It is a **dependent system**.

Determining the Type of System Algebraically

A third option for identifying systems as consistent, inconsistent or dependent is to just solve the system and use the result as a guide.

Example 4

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}10x - 3y &= 3 \\2x + y &= 9\end{aligned}$$

Solution

Let's solve this system using the substitution method.

Solve the second equation for y :

$$2x + y = 9 \Rightarrow y = -2x + 9$$

Substitute that expression for y in the first equation:

$$\begin{aligned}10x - 3y &= 3 \\10x - 3(-2x + 9) &= 3 \\10x + 6x - 27 &= 3 \\16x &= 30 \\x &= \frac{15}{8}\end{aligned}$$

Substitute the value of x back into the second equation and solve for y :

$$2x + y = 9 \Rightarrow y = -2x + 9 \Rightarrow y = -2 \cdot \frac{15}{8} + 9 \Rightarrow y = \frac{21}{4}$$

The solution to the system is $(\frac{15}{8}, \frac{21}{4})$. The system is **consistent** since it has only one solution.

Example 5

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}3x - 2y &= 4 \\9x - 6y &= 1\end{aligned}$$

Solution

Let's solve this system by the method of multiplication.

Multiply the first equation by 3:

$$\begin{array}{rcl}3(3x - 2y = 4) & & 9x - 6y = 12 \\ & \Rightarrow & \\9x - 6y = 1 & & 9x - 6y = 1\end{array}$$

Add the two equations:

$$\begin{array}{r}9x - 6y = 4 \\9x - 6y = 1 \\ \hline0 = 13\end{array} \quad \text{This statement is not true.}$$

If our solution to a system turns out to be a statement that is not true, then the system doesn't really have a solution; it is **inconsistent**.

Example 6

Solve the following system of equations. Identify the system as consistent, inconsistent or dependent.

$$\begin{aligned}4x + y &= 3 \\12x + 3y &= 9\end{aligned}$$

Solution

Let's solve this system by substitution.

Solve the first equation for y :

$$4x + y = 3 \Rightarrow y = -4x + 3$$

Substitute this expression for y in the second equation:

$$\begin{aligned}12x + 3y &= 9 \\12x + 3(-4x + 3) &= 9 \\12x - 12x + 9 &= 9 \\9 &= 9\end{aligned}$$

This statement is always true.

If our solution to a system turns out to be a statement that is always true, then the system is **dependent**.

A second glance at the system in this example reveals that the second equation is three times the first equation, so the two lines are identical. The system has an infinite number of solutions because they are really the same equation and trace out the same line.

Let's clarify this statement. An infinite number of solutions does not mean that *any* ordered pair (x, y) satisfies the system of equations. Only ordered pairs that solve the equation in the system (either one of the equations) are also solutions to the system. There are infinitely many of these solutions to the system because there are infinitely many points on any one line.

For example, $(1, -1)$ is a solution to the system in this example, and so is $(-1, 7)$. Each of them fits both the equations because both equations are really the same equation. But $(3, 5)$ doesn't fit either equation and is not a solution to the system.

In fact, for every x -value there is just one y -value that fits both equations, and for every y -value there is exactly one x -value—just as there is for a single line.

Let's summarize how to determine the type of system we are dealing with algebraically.

- A **consistent system** will always give exactly one solution.
- An **inconsistent system** will yield a statement that is *always false* (like $0 = 13$).
- A **dependent system** will yield a statement that is *always true* (like $9 = 9$).

Applications

In this section, we'll see how consistent, inconsistent and dependent systems might arise in real life.

Example 7

The movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of \$45 and then rent each movie for \$2 or they can choose not to pay the membership fee and rent each movie for \$3.50. How many movies would you have to rent before the membership becomes the cheaper option?

Solution

Let's translate this problem into algebra. Since there are two different options to consider, we can write two different equations and form a system.

The choices are "membership" and "no membership." We'll call the number of movies you rent x and the total cost of renting movies for a year y .

Table 7.7:

	flat fee	rental fee	total
membership	\$45	$2x$	$y = 45 + 2x$
no membership	\$0	$3.50x$	$y = 3.5x$

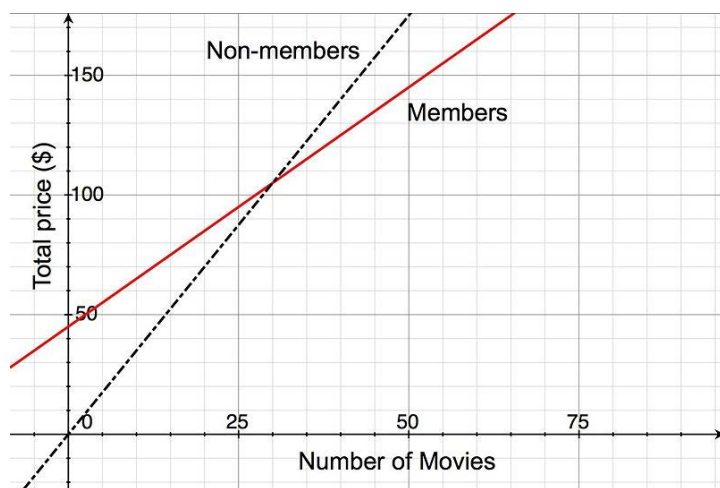
The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent a movie. For the membership option the rental fee is $2x$, since you would pay \$2 for each movie you rented; for the no membership option the rental fee is $3.50x$, since you would pay \$3.50 for each movie you rented.

Our system of equations is:

$$y = 45 + 2x$$

$$y = 3.50x$$

Here's a graph of the system:



Now we need to find the exact intersection point. Since each equation is already solved for y , we can easily solve the system with substitution. Substitute the second equation into the first one:

$$y = 45 + 2x$$

$$\Rightarrow 3.50x = 45 + 2x \Rightarrow 1.50x = 45 \Rightarrow x = 30 \text{ movies}$$

$$y = 3.50x$$

You would have to rent **30 movies per year** before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem

Now let's look at a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (and the y -intercept is different). Let's change the previous problem so that this is the case.

Example 8

Two movie rental stores are in competition. Movie House charges an annual membership of \$30 and charges \$3 per movie rental. Flicks for Cheap charges an annual membership of \$15 and charges \$3 per

movie rental. After how many movie rentals would Movie House become the better option?

Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines on a graph that describe each option have different y -intercepts—namely 30 for Movie House and 15 for Flicks for Cheap—but the same slope: 3 dollars per movie. This means that the lines are parallel and so the system is inconsistent.

Now let's see how this works algebraically. Once again, we'll call the number of movies you rent x and the total cost of renting movies for a year y .

Table 7.8:

	flat fee	rental fee	total
Movie House	\$30	$3x$	$y = 30 + 3x$
Flicks for Cheap	\$15	$3x$	$y = 15 + 3x$

The system of equations that describes this problem is:

$$y = 30 + 3x$$

$$y = 15 + 3x$$

Let's solve this system by substituting the second equation into the first equation:

$$y = 30 + 3x$$

$$\Rightarrow 15 + 3x = 30 + 3x \Rightarrow 15 = 30 \quad \text{This statement is always false.}$$

$$y = 15 + 3x$$

This means that the system is **inconsistent**.

Example 9

Peter buys two apples and three bananas for \$4. Nadia buys four apples and six bananas for \$8 from the same store. How much does one banana and one apple cost?

Solution

We must write two equations: one for Peter's purchase and one for Nadia's purchase.

Let's say a is the cost of one apple and b is the cost of one banana.

Table 7.9:

	cost of apples	cost of bananas	total cost
Peter	$2a$	$3b$	$2a + 3b = 4$
Nadia	$4a$	$6b$	$4a + 6b = 8$

The system of equations that describes this problem is:

$$2a + 3b = 4$$

$$4a + 6b = 8$$

Let's solve this system by multiplying the first equation by -2 and adding the two equations:

$$-2(2a + 3b = 4)$$

$$-4a - 6b = -8$$

$$4a + 6b = 8 \quad \Rightarrow \quad \frac{4a + 6b = 8}{0 + 0 = 0}$$

This statement is always true. This means that the system is **dependent**.

Looking at the problem again, we can see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for \$4, it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price (\$8). Since the second equation doesn't give us any new information, it doesn't make it possible to find out the price of each fruit.

Review Questions

Express each equation in slope-intercept form. Without graphing, state whether the system of equations is consistent, inconsistent or dependent.

1. $3x - 4y = 13$
 $y = -3x - 7$
2. $\frac{3}{5}x + y = 3$
 $1.2x + 2y = 6$
3. $3x - 4y = 13$
 $y = -3x - 7$
4. $3x - 3y = 3$
 $x - y = 1$
5. $0.5x - y = 30$
 $0.5x - y = -30$
6. $4x - 2y = -2$
 $3x + 2y = -12$
7. $3x + y = 4$
 $y = 5 - 3x$
8. $x - 2y = 7$
 $4y - 2x = 14$

Find the solution of each system of equations using the method of your choice. State if the system is inconsistent or dependent.

9. $3x + 2y = 4$
 $-2x + 2y = 24$
10. $5x - 2y = 3$
 $2x - 3y = 10$
11. $3x - 4y = 13$
 $y = -3x - 7$
12. $5x - 4y = 1$
 $-10x + 8y = -30$
13. $4x + 5y = 0$
 $3x = 6y + 4.5$
14. $-2y + 4x = 8$
 $y - 2x = -4$
15. $x - \frac{1}{2}y = \frac{3}{2}$
 $3x + y = 6$

16. $0.05x + 0.25y = 6$
 $x + y = 24$
17. $x + \frac{2}{3}y = 6$
 $3x + 2y = 2$
18. A movie theater charges \$4.50 for children and \$8.00 for adults.
- On a certain day, 1200 people enter the theater and \$8375 is collected. How many children and how many adults attended?
 - The next day, the manager announces that she wants to see them take in \$10000 in tickets. If there are 240 seats in the house and only five movie showings planned that day, is it possible to meet that goal?
 - At the same theater, a 16-ounce soda costs \$3 and a 32-ounce soda costs \$5. If the theater sells 12,480 ounces of soda for \$2100, how many people bought soda? (**Note:** Be careful in setting up this problem!)
19. Jamal placed two orders with an internet clothing store. The first order was for 13 ties and 4 pairs of suspenders, and totaled \$487. The second order was for 6 ties and 2 pairs of suspenders, and totaled \$232. The bill does not list the per-item price, but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
20. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane's speed in still air and the jet-stream's speed?
21. Nadia told Peter that she went to the farmer's market and bought two apples and one banana, and that it cost her \$2.50. She thought that Peter might like some fruit, so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so he would only pay her for four apples. Nadia told him that the second time she paid \$6.00 for the fruit.
- What did Peter find when he tried to figure out the price of four apples?
 - Nadia then told Peter she had made a mistake, and she actually paid \$5.00 on her second trip. Now what answer did Peter get when he tried to figure out how much to pay her?
 - Alicia then showed up and told them she had just bought 3 apples and 2 bananas from the same seller for \$4.25. Now how much should Peter pay Nadia for four apples?

7.5 Systems of Linear Inequalities

Learning Objectives

- Graph linear inequalities in two variables.
- Solve systems of linear inequalities.
- Solve optimization problems.

Introduction

In the last chapter you learned how to graph a linear inequality in two variables. To do that, you graphed the equation of the straight line on the coordinate plane. The line was solid for \leq or \geq signs (where the equals sign is included), and the line was dashed for $<$ or $>$ signs (where the equals sign is not included). Then you shaded above the line (if the inequality began with $y >$ or $y \geq$) or below the line (if it began with $y <$ or $y \leq$).

In this section, we'll see how to graph two or more linear inequalities on the same coordinate plane. The inequalities are graphed separately on the same graph, and the solution for the system is the common

shaded region between all the inequalities in the system. One linear inequality in two variables divides the plane into two **half-planes**. A **system** of two or more linear inequalities can divide the plane into more complex shapes.

Let's start by solving a system of two inequalities.

Graph a System of Two Linear Inequalities

Example 1

Solve the following system:

$$2x + 3y \leq 18$$

$$x - 4y \leq 12$$

Solution

Solving systems of linear inequalities means graphing and finding the intersections. So we graph each inequality, and then find the intersection *regions* of the solution.

First, let's rewrite each equation in slope-intercept form. (Remember that this form makes it easier to tell which region of the coordinate plane to shade.) Our system becomes

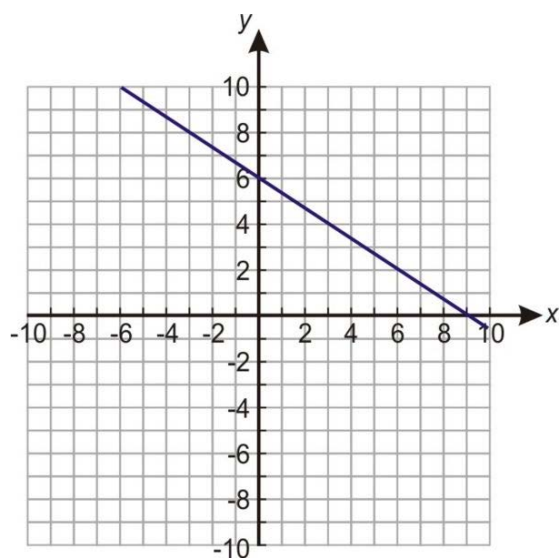
$$3y \leq -2x + 18 \quad \Rightarrow \quad y \leq -\frac{2}{3}x + 6$$

$$-4y \leq -x + 12 \quad \Rightarrow \quad y \geq \frac{x}{4} - 3$$

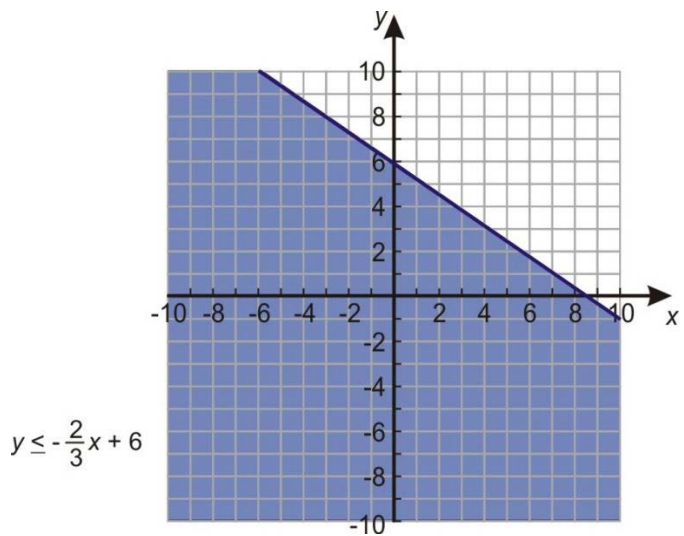
Notice that the inequality sign in the second equation changed because we divided by a negative number!

For this first example, we'll graph each inequality separately and then combine the results.

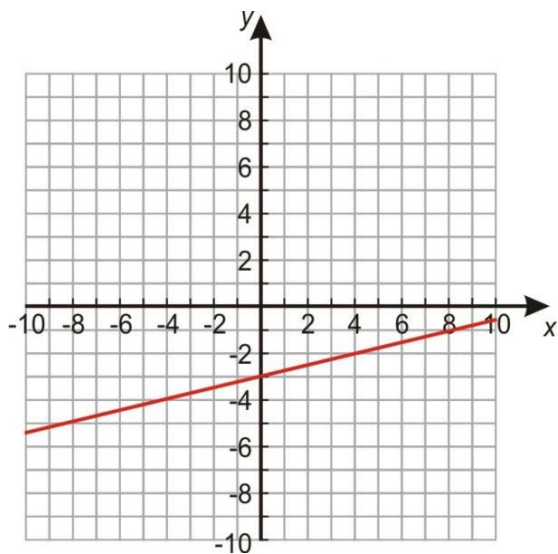
Here's the graph of the first inequality:



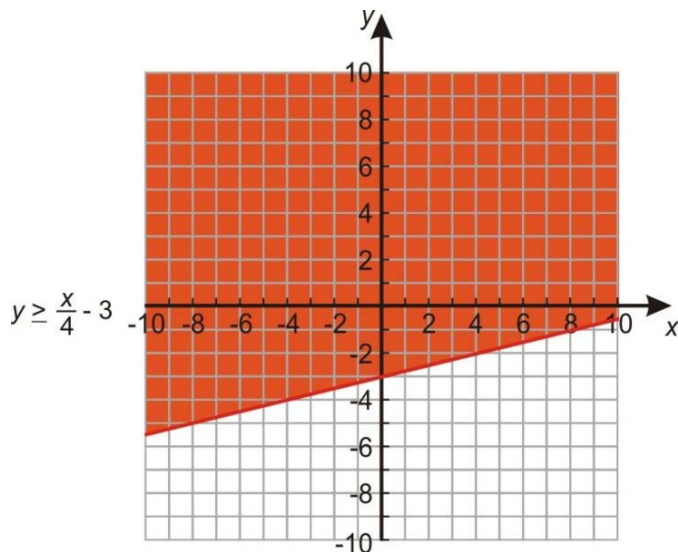
The line is solid because the equals sign is included in the inequality. Since the inequality is **less than or equal to**, we shade **below** the line.



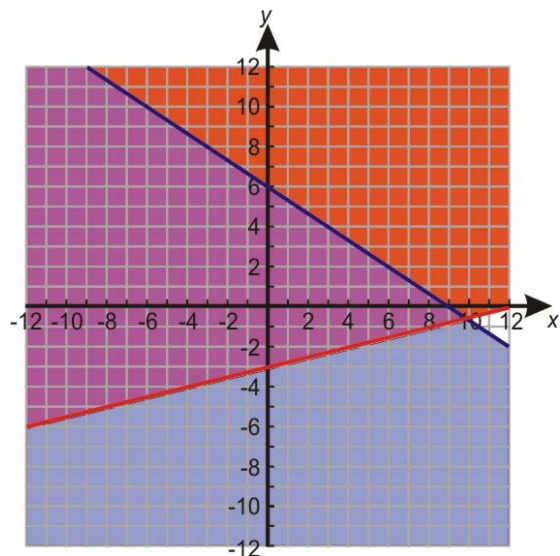
And here's the graph of the second inequality:



The line is solid again because the equals sign is included in the inequality. We now shade **above** the line because y is **greater** than or equal to.



When we combine the graphs, we see that the blue and red shaded regions overlap. The area where they overlap is the area where both inequalities are true. Thus that area (shown below in purple) is the solution of the system.



The kind of solution displayed in this example is called **unbounded**, because it continues forever in at least one direction (in this case, forever upward and to the left).

Example 2

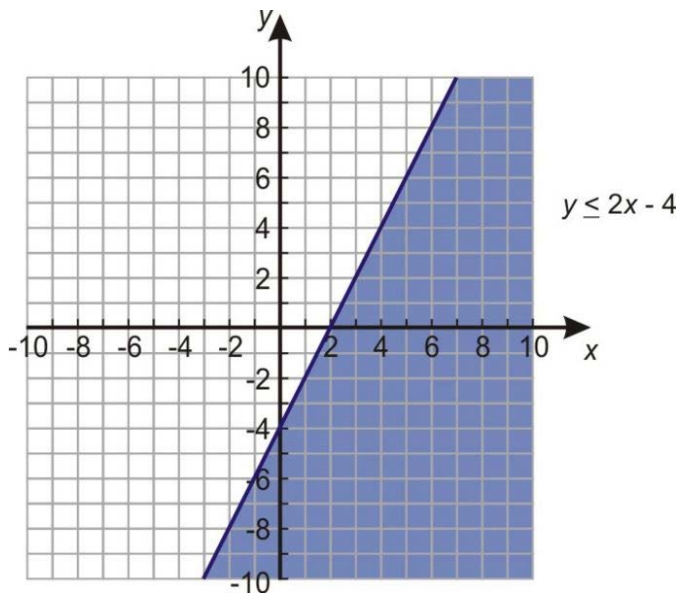
There are also situations where a system of inequalities has no solution. For example, let's solve this system.

$$y \leq 2x - 4$$

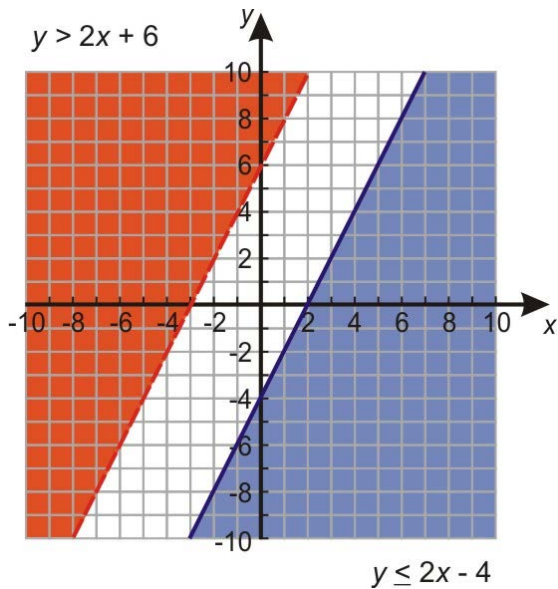
$$y > 2x + 6$$

Solution

We start by graphing the first line. The line will be solid because the equals sign is included in the inequality. We must shade downwards because y is less than.



Next we graph the second line on the same coordinate axis. This line will be dashed because the equals sign is not included in the inequality. We must shade upward because y is greater than.



It doesn't look like the two shaded regions overlap at all. The two lines have the same slope, so we know they are parallel; that means that the regions indeed won't ever overlap since the lines won't ever cross. So this system of inequalities has no solution.

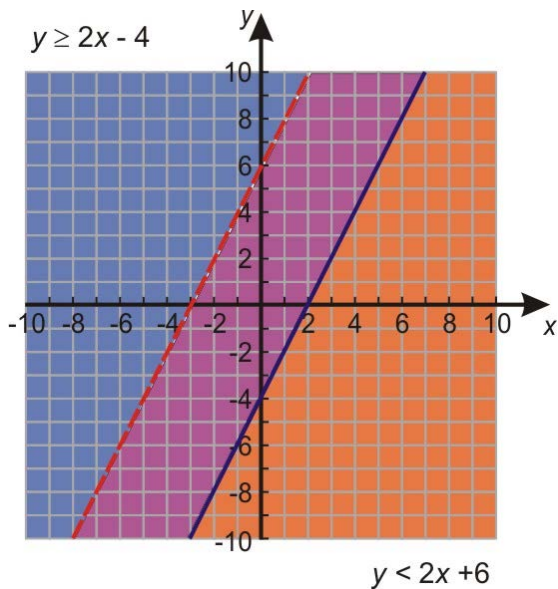
But a system of inequalities can sometimes have a solution even if the lines are parallel. For example, what happens if we swap the directions of the inequality signs in the system we just graphed?

To graph the system

$$y \geq 2x - 4,$$

$$y < 2x + 6,$$

we draw the same lines we drew for the previous system, but we shade *upward* for the first inequality and *downward* for the second inequality. Here is the result:



You can see that this time the shaded regions overlap. The area between the two lines is the solution to the system.

Graph a System of More Than Two Linear Inequalities

When we solve a system of just two linear inequalities, the solution is always an **unbounded** region—one that continues infinitely in at least one direction. But if we put together a system of more than two inequalities, sometimes we can get a solution that is **bounded**—a finite region with three or more sides.

Let's look at a simple example.

Example 3

Find the solution to the following system of inequalities.

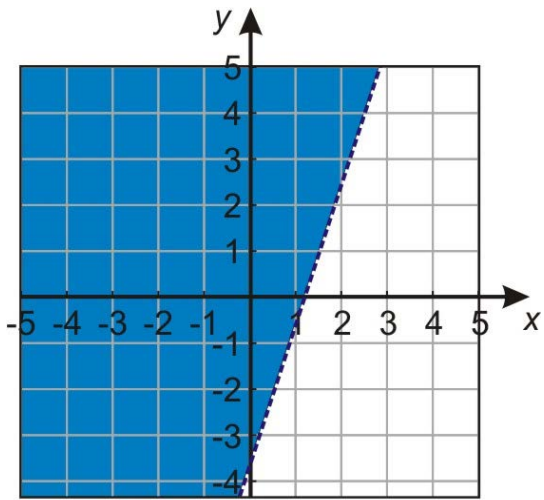
$$\begin{aligned} 3x - y &< 4 \\ 4y + 9x &< 8 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution

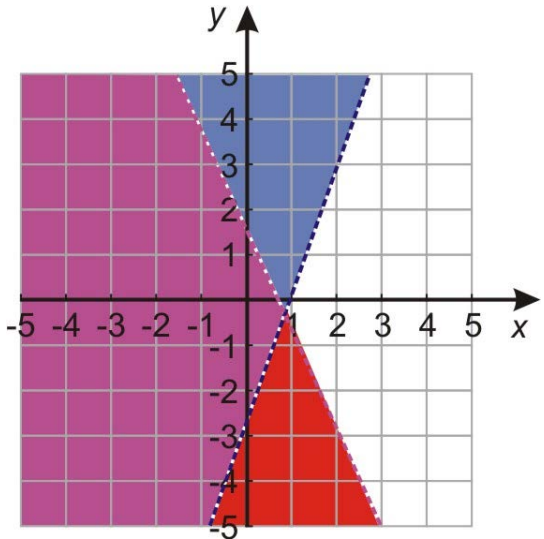
Let's start by writing our inequalities in slope-intercept form.

$$\begin{aligned} y &> 3x - 4 \\ y &< -\frac{9}{4}x + 2 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

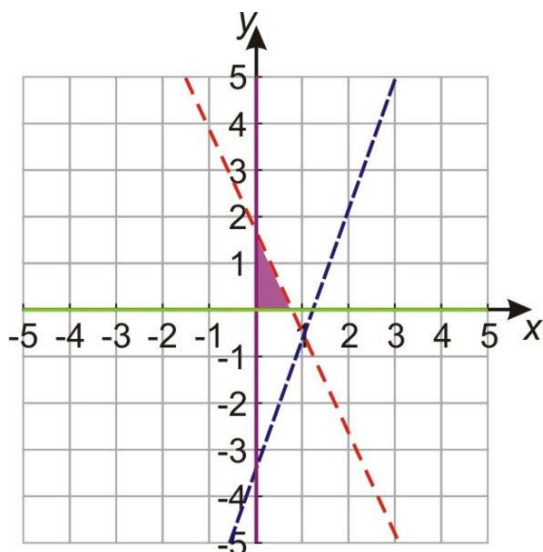
Now we can graph each line and shade appropriately. First we graph $y > 3x - 4$:



Next we graph $y < -\frac{9}{4}x + 2$:



Finally we graph $x \geq 0$ and $y \geq 0$, and we're left with the region below; this is where all four inequalities overlap.



The solution is **bounded** because there are lines on all sides of the solution region. In other words, the solution region is a bounded geometric figure, in this case a triangle.

Notice, too, that only three of the lines we graphed actually form the boundaries of the region. Sometimes when we graph multiple inequalities, it turns out that some of them don't affect the overall solution; in this case, the solution would be the same even if we'd left out the inequality $y > 3x - 4$. That's because the solution region of the system formed by the other three inequalities is completely contained within the solution region of that fourth inequality; in other words, any solution to the other three inequalities is *automatically* a solution to that one too, so adding that inequality doesn't narrow down the solution set at all.

But that wasn't obvious until we actually drew the graph!

Solve Real-World Problems Using Systems of Linear Inequalities

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called **linear programming** problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the *best* possible value under those conditions. A typical example would be taking the limitations of materials and labor at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we'll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called **constraints**) to form a bounded area on the coordinate plane (called

the **feasibility region**).

- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the **optimization equation**) for which you're trying to find the **maximum** or **minimum** value.

Example 4

If $z = 2x + 5y$, find the maximum and minimum values of z given these constraints:

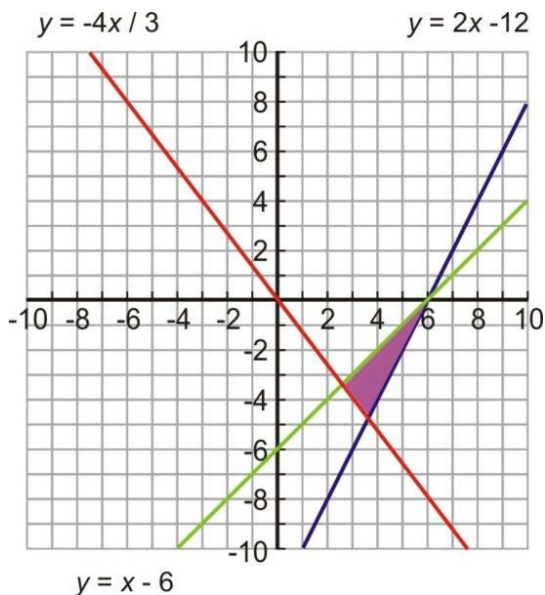
$$\begin{aligned}2x - y &\leq 12 \\ 4x + 3y &\geq 0 \\ x - y &\leq 6\end{aligned}$$

Solution

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$\begin{aligned}y &\geq 2x - 12 \\ y &\geq -\frac{4}{3}x \\ y &\geq x - 6\end{aligned}$$

These three linear inequalities are called the **constraints**, and here is their graph:



The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable z within that region. In other words, which values of x and y within the feasibility region will give us the greatest and smallest overall values for the expression $2x + 5y$?

Fortunately, we don't have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out *which* vertices. So

for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of z at that point.

The first system consists of the equations $y = 2x - 12$ and $y = -\frac{4}{3}x$. We can solve this system by substitution:

$$\begin{aligned}-\frac{4}{3}x &= 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6 \\ y &= 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8\end{aligned}$$

The lines intersect at the point $(3.6, -4.8)$.

The second system consists of the equations $y = 2x - 12$ and $y = x - 6$. Solving this system by substitution:

$$\begin{aligned}x - 6 &= 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6 \\ y &= x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 6\end{aligned}$$

The lines intersect at the point $(6, 6)$.

The third system consists of the equations $y = -\frac{4}{3}x$ and $y = x - 6$. Solving this system by substitution:

$$\begin{aligned}x - 6 &= -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57 \\ y &= x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43\end{aligned}$$

The lines intersect at the point $(2.57, -3.43)$.

So now we have three different points that might give us the maximum and minimum values for z . To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation $z = 2x + 5y$.

When we plug in $(3.6, -4.8)$, we get $z = 2(3.6) + 5(-4.8) = -16.8$.

When we plug in $(6, 0)$, we get $z = 2(6) + 5(0) = 12$.

When we plug in $(2.57, -3.43)$, we get $z = 2(2.57) + 5(-3.43) = -12.01$.

So we can see that **the point $(6, 0)$ gives us the maximum possible value for z and the point $(3.6, -4.8)$ gives us the minimum value.**

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we'll look at a real-life application.

Example 5

You have \$10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank's CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than \$1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What's the best way to distribute your money given these constraints?

Solution

Let's define our variables:

x is the amount of money invested in the municipal bond at 5% return

y is the amount of money invested in the bank's CD at 7% return

$10000 - x - y$ is the amount of money invested in the high-risk account at 10% return

z is the total interest returned from all the investments, so $z = .05x + .07y + .1(10000 - x - y)$ or $z = 1000 - 0.05x - 0.03y$. This is the amount that we are trying to maximize. Our goal is to find the values of x and y that maximizes the value of z .

Now, let's write inequalities for the *constraints*:

You decide not to invest more than \$1000 in the high-risk account—that means:

$$10000 - x - y \leq 1000$$

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

$$3y \leq x$$

Also, you can't invest less than zero dollars in each account, so:

$$x \geq 0$$

$$y \geq 0$$

$$10000 - x - y \geq 0$$

To summarize, we must maximize the expression $z = 1000 - .05x - .03y$ using the constraints:

$$10000 - x - y \leq 1000$$

$$3y \leq x$$

$$x \geq 0$$

$$y \geq 0$$

$$10000 - x - y \geq 0$$

$$y \geq 9000 - x$$

$$y \leq \frac{x}{3}$$

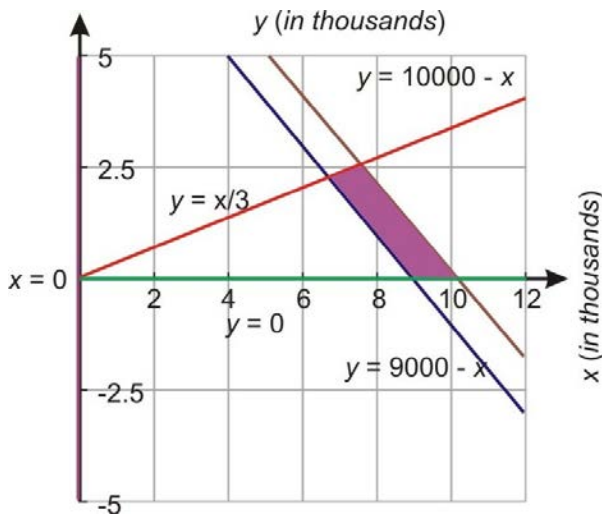
$$x \geq 0$$

$$y \geq 0$$

$$y \leq 10000 - x$$

Or in slope-intercept form:

Step 1: Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:



The purple region is the feasibility region where all the possible solutions can occur.

Step 2: Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

System 1:

$$y = \frac{x}{3}$$

$$y = 10000 - x$$

Substitute the first equation into the second equation:

$$\begin{aligned}\frac{x}{3} &= 10000 - x \Rightarrow x = 30000 - 3x \Rightarrow 4x = 30000 \Rightarrow x = 7500 \\ y &= \frac{x}{3} \Rightarrow y = \frac{7500}{3} \Rightarrow y = 2500\end{aligned}$$

The intersection point is (7500, 2500).

System 2:

$$\begin{aligned}y &= \frac{x}{3} \\ y &= 9000 - x\end{aligned}$$

Substitute the first equation into the second equation:

$$\begin{aligned}\frac{x}{3} &= 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750 \\ y &= \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250\end{aligned}$$

The intersection point is (6750, 2250).

System 3:

$$\begin{aligned}y &= 0 \\ y &= 10000 - x.\end{aligned}$$

The intersection point is (10000, 0).

System 4:

$$\begin{aligned}y &= 0 \\ y &= 9000 - x.\end{aligned}$$

The intersection point is (9000, 0).

Step 3: In order to find the maximum value for z , we need to plug all the intersection points into the equation for z and find which one yields the largest number.

$$(7500, 2500): z = 1000 - 0.05(7500) - 0.03(2500) = 550$$

$$(6750, 2250): z = 1000 - 0.05(6750) - 0.03(2250) = 595$$

$$(10000, 0): z = 1000 - 0.05(10000) - 0.03(0) = 500$$

$$(9000, 0): z = 1000 - 0.05(9000) - 0.03(0) = 550$$

The maximum return on the investment of \$595 occurs at the point (6750, 2250). This means that:

\$6,750 is invested in the municipal bonds.

\$2,250 is invested in the bank CDs.

\$1,000 is invested in the high-risk account.

Graphing calculators can be very useful for problems that involve this many inequalities. The video at http://www.youtube.com/watch?v=__wAxkYmhvY shows a real-world linear programming problem worked through in detail on a graphing calculator, although the methods used there can also be used for pencil-and-paper solving.

Review Questions

1. Consider the system

$$y < 3x - 5$$

$$y > 3x - 5.$$

Is it consistent or inconsistent? Why?

2. Consider the system

$$y \leq 2x + 3$$

$$y \geq 2x + 3.$$

Is it consistent or inconsistent? Why?

3. Consider the system

$$y \leq -x + 1$$

$$y > -x + 1.$$

Is it consistent or inconsistent? Why?

4. In example 3 in this lesson, we solved a system of four inequalities and saw that one of the inequalities, $y > 3x - 4$, didn't affect the solution set of the system.

(a) What would happen if we changed that inequality to $y < 3x - 4$?

(b) What's another inequality that we could add to the original system without changing it? Show how by sketching a graph of that inequality along with the rest of the system.

(c) What's another inequality that we could add to the original system to make it inconsistent? Show how by sketching a graph of that inequality along with the rest of the system.

5. Recall the compound inequalities in one variable that we worked with back in chapter 6. Compound inequalities with "and" are simply systems like the ones we are working with here, except with one variable instead of two.

(a) Graph the inequality $x > 3$ in two dimensions. What's another inequality that could be combined with it to make an inconsistent system?

(b) Graph the inequality $x \leq 4$ on a number line. What two-dimensional system would have a graph that looks just like this one?

Find the solution region of the following systems of inequalities.

6. $x - y < -6$

$$2y \geq 3x + 17$$

7. $4y - 5x < 8$

$$-5x \geq 16 - 8y$$

8. $5x - y \geq 5$

$$2y - x \geq -10$$

9. $5x + 2y \geq -25$

$$3x - 2y \leq 17$$

$$x - 6y \geq 27$$

10. $2x - 3y \leq 21$

$$x + 4y \leq 6$$

$$3x + y \geq -4$$

11. $12x - 7y < 120$

$$7x - 8y \geq 36$$

$$5x + y \geq 12$$

Solve the following linear programming problems.

12. Given the following constraints, find the maximum and minimum values of $z = -x + 5y$:

$$x + 3y \leq 0$$

$$x - y \geq 0$$

$$3x - 7y \leq 16$$

13. Santa Claus is assigning elves to work an eight-hour shift making toy trucks. Apprentice elves draw a wage of five candy canes per hour worked, but can only make four trucks an hour. Senior elves can make six trucks an hour and are paid eight candy canes per hour. There's only room for nine elves in the truck shop, and due to a candy-makers' strike, Santa Claus can only pay out 480 candy canes for the whole 8-hour shift.
- How many senior elves and how many apprentice elves should work this shift to maximize the number of trucks that get made?
 - How many trucks will be made?
 - Just before the shift begins, the apprentice elves demand a wage increase; they insist on being paid seven candy canes an hour. Now how many apprentice elves and how many senior elves should Santa assign to this shift?
 - How many trucks will now get made, and how many candy canes will Santa have left over?
14. In Adrian's Furniture Shop, Adrian assembles both bookcases and TV cabinets. Each type of furniture takes her about the same time to assemble. She figures she has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost her \$20 and the materials for each TV stand costs her \$45. She has \$600 to spend on materials. Adrian makes a profit of \$60 on each bookcase and a profit of \$100 on each TV stand.
- Set up a system of inequalities. What x - and y -values do you get for the point where Adrian's profit is maximized? Does this solution make sense in the real world?
 - What two possible real-world x -values and what two possible real-world y -values would be closest to the values in that solution?
 - With two choices each for x and y , there are four possible combinations of x - and y -values. Of those four combinations, which ones actually fall within the feasibility region of the problem?
 - Which one of those feasible combinations seems like it would generate the most profit? Test out each one to confirm your guess. How much profit will Adrian make with that combination?
 - Based on Adrian's previous sales figures, she doesn't think she can sell more than 8 TV stands. Now how many of each piece of furniture should she make, and what will her profit be?
 - Suppose Adrian is confident she can sell all the furniture she can make, but she doesn't have room to display more than 7 bookcases in her shop. Now how many of each piece of furniture should she make, and what will her profit be?
15. Here's a "linear programming" problem on a line instead of a plane: Given the constraints $x \leq 5$ and $x \geq -2$, maximize the value of y where $y = x + 3$.

Texas Instruments Resources

In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See <http://www.ck12.org/flexr/chapter/9617>.